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Faculty of Science and Technology
Mathematics and Computer Science Department

Linear Algebra Courses and exercises

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Forword

This handout is a resource on key Linear algebra concepts, focusing on several fundamental topics in mathematical algebra and the theory of matrices.

This handout is intended for first-year Computer Science student. It is also a valuable reference for first-year science and technology students, and for anyone wishing to become familiar with basic mathematical methods in linear algebra and matrix theory.

I have endeavored to write the chapters of this handout in an accessible way, concentrating on the important themes of the curricula.

Firstly, I wanted this handout to be directly usable by a first-year Computer Science student.

The author hopes that this document, despite its imperfections, will be useful to students and readers. It was written and published for this purpose.

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Detailed Module Syllabus : Mathematics-II

Teaching Unit : Fundamental

Teaching Objectives : Introduction to the basic principles of vector spaces.

Recommended prior knowledge basic algebra concepts.

Chapter-1 Vector Space

- 1.1- Definition
- 1.2- Vector Subspace, Examples
- 1.3- Linearly Independent Families, Spanning Families. Bases, Dimension
- 1.4- Finite-Dimensional Vector Space (Properties)
- 1.5- Sum and Direct sums of subspaces

Chapter-2 Linear Maps

- 2.1- Definition
- 2.2- Image and Kernel of a Linear Map
- 2.3- Rank of a Linear Map, Rank Theorem
- 2.4- Composition of Linear Maps. Inverse of a Bijective Linear Map, Automorphism

Chapter-3 Matrices

- 3.1- Matrix Associated with a Linear Map
- 3.2- Operations on Matrices : Sum, Product of Two Matrices, Transpose Matrix
- 3.3- Vector Space of Matrices with n Rows and m Columns
- 3.3- Ring of Square Matrices. Determinant of a Square Matrix and Properties. Invertible Matrices
- 3.4- Rank of a Matrix (Associated Linear Map). Invariance of Rank under Transposition

Chapter-3 Solving Systems of Equations

- 4.1- System of Equations, Matrix Representation
- 4.2- Rank of a System of Equations
- 4.3- Cramer's Rule

Vector Spaces

[Cherney] Linear algebra is the study of vectors and linear functions.

Many difficult problems can be handled easily once relevant information is organized in a certain way. This text aims to teach you how to organize information in cases where certain mathematical structures are present. Linear algebra is, in general, the study of those structures. Namely.

In broad terms, vectors are things you can add and linear functions are functions of vectors that respect vector addition. The goal of this text is to teach you to organize information about vector spaces in a way that makes problems involving linear functions of many variables easy. (Or at least tractable.)

To get a feel for the general idea of organizing information, of vectors, and of linear functions this chapter has brief sections on each. We start here in hopes of putting students in the right mindset for the odyssey that follows; the latter chapters cover the same material at a slower pace. Please be prepared to change the way you think about some familiar mathematical objects and keep a pencil and piece of paper handy!

1.1 Vector Spaces

Definition 1.1.1. Let $(\mathbb{F}, +, \cdot)$ be a field, and \mathbb{V} be a set of objects on which two operations are defined :

$$* : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$$

and scalar multiplication

$$\odot : \mathbb{F} \times \mathbb{V} \longrightarrow \mathbb{V}$$

\mathbb{V} is called a vector space over a field $\mathbb{V}(\mathbb{F})$ if all the following are satisfied :

- 1- $(\mathbb{V}, *)$ is an Abelian group (commutative group)
- 2- $\forall u, v \in \mathbb{V}, \forall \alpha, \beta \in \mathbb{F}$

a- $\alpha \odot (u * v) = \alpha \odot u * \alpha \odot v$

b- $(\alpha + \beta) \odot u = \alpha \odot u * \beta \odot u$

c- $\alpha \odot (\beta \odot u) = (\alpha \cdot \beta) \odot u$

$$\mathbf{d-} \quad 1_{\mathbb{F}} \odot u = u$$

Example 1. Let $+$ and \cdot be a standard operations.

1. $(\mathbb{R}, +, \cdot)$ is a \mathbb{R} -vector space,
2. $(\mathbb{C}, +, \cdot)$ is a \mathbb{R} -vector space,
3. $(\mathbb{C}, +, \cdot)$ is a \mathbb{C} -vector space,
4. $(\mathbb{R}^n, +, \cdot)$ is a \mathbb{R} -vector space, ($n \in \mathbb{N}^*$),
5. Let $n \geq 0$ be an integer and let $P_n = \mathbb{R}_n[X]$ the set of all polynomials of degree at most n . Members of P_n have the form

$$p(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$$

where a_0, a_1, \dots, a_n are real numbers and t is a real variable. The set $P_n = \mathbb{R}_n[X]$ is a real vector space.

6. Let $\mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{R}; \mathbb{R})$ be a set of all functions from \mathbb{R} to \mathbb{R} , define addition and scalar multiplication of functions by :

$$\forall f, g \in \mathcal{F}(\mathbb{R}), \forall \alpha \in \mathbb{R}; (f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x)$$

The function space $\mathcal{F}(\mathbb{R})$ is real vector space.

Example 2. Check whether the set $V = \{x, y, z \in \mathbb{R} : 2 + 3y^3 - 4z^2 = 0\}$ is vector space or not ?

Solution : Given $V = \{x, y, z \in \mathbb{R} : 2 + 3y^3 - 4z^2 = 0\}$.

So, take $u_1 = (2, 0, 1), u_2 = (8, 0, 2) \in V$, but $u_1 + u_2 = (10, 0, 3) \notin V$

Because $2(10) + 3(0)^3 - 4(2)^2 = -36 \neq 0$. i.e. additive vector $(10, 0, 3)$ does not belongs to V .

Therefore. set V is not a vector space.

Exercise 1. Check whether the set $V = \{(x, y, z) \in \mathbb{R}^3 : (x - y)z = 0\}$ is vector space or not ?

Solution 1. Given $V = \{(x, y, z) \in \mathbb{R}^3 : (x - y)z = 0\}$.

So, take $u_1 = (1, 1, 2), u_2 = (2, 5, 0) \in V$, but $u_1 + u_2 = (3, 6, 2) \notin V$

Because $(3 - 6)2 = -6 \neq 0$. i.e. the closure property is not satisfied .

Therefore. set V is not a vector space.

Exercise 2. Show that \mathbb{R}^2 is not vector space over \mathbb{R} under the addition and scalar multiplication defined by :

$$\forall (x, y), (x', y') \in \mathbb{R}^2; \forall \alpha \in \mathbb{R} : (x, y) \oplus (x', y') = (0, y + y'), \quad \alpha \odot (x, y) = (\alpha x, \alpha y)$$

Solution 2. \mathbb{R}^2 is not vector space under the compositions \oplus and \odot , because

$$(\alpha + \beta) \odot (x, y) = ((\alpha + \beta)x, (\alpha + \beta)y), \quad \alpha \odot (x, y) \oplus \beta \odot (x, y) = (0, \alpha y + \beta y)$$

and

$$(\alpha + \beta) \odot (x, y) \neq \alpha \odot (x, y) \oplus \beta \odot (x, y)$$

1.1.1 Subspaces

Vector spaces may be formed from subsets of other vectors spaces. These are called subspaces.

Definition 1.1.2. Let \mathbb{V} be a vector space over a field \mathbb{F} . If $\mathbb{W} \subset \mathbb{V}$ such that \mathbb{W} is a vector space over \mathbb{F} with respect to the operations of \mathbb{V} restricted to \mathbb{W} then we say \mathbb{W} is a subspace of \mathbb{V} and write $W \leq V$.

Theorem 1.1.1. A subspace of a vector space \mathbb{V} is a subset \mathbb{H} of \mathbb{V} that has three properties :

1. The zero vector of \mathbb{V} is in \mathbb{H} .

$$0_{\mathbb{V}} \in \mathbb{H}$$

2. For each u and v are in \mathbb{H} , $u + v$ is in \mathbb{H} . (In this case we say \mathbb{H} is closed under vector addition.)

$$\forall u, v \in \mathbb{H}, u \oplus v \in \mathbb{H}$$

3. For each u in \mathbb{H} and each scalar c , cu is in \mathbb{H} . (In this case we say \mathbb{H} is closed under scalar multiplication.)

$$\forall u \in \mathbb{H}, \forall \alpha \in \mathbb{K}, \alpha \odot u \in \mathbb{H}$$

If the subset \mathbb{H} satisfies these three properties, then \mathbb{H} itself is a **vector space**.

Recap

1. To show that \mathbb{H} is a subspace of a vector space, use **Theorem 1.1.1**
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms 1, 2 or 3 (from the definition of a subspace) is violated.

Example 3.

1. $\{0_{\mathbb{V}}\}$ is called the trivial subspace. and \mathbb{V} is the subspace of \mathbb{V} .
2. If $\mathbb{V} = \mathbb{R}^3$ then :
 - $\{(0, 0, 0)\}$ is a subspace,
 - any line through the origin is a subspace,
 - any plane through the origin is a subspace.
3. The function space $\mathcal{F}(\mathbb{R})$ has many subspaces.
 - continuous functions : $\mathcal{C}(\mathbb{R})$
 - differentiable functions : $\mathcal{C}^1(\mathbb{R})$
 - smooth functions : $\mathcal{C}^\infty(\mathbb{R})$
 - polynomial functions (which are naturally identified with $\mathbb{R}[x]$)

Example 4. Let $\mathbb{H} = \{(x, 0, z) : x, y \in \mathbb{R}\}$. Show that \mathbb{H} is a subspace of \mathbb{R}^3 .

Solution : Verify properties of the definition of a subspace.

1. The zero vector of $V = \mathbb{R}^3$ is in \mathbb{H} (let $x = z = 0$).
2. Adding two vectors in \mathbb{H} always produces another vector whose second entry is and therefore the sum of two vectors in \mathbb{H} is also in \mathbb{H} .

$$(x, 0, z) + (x', 0, z') = (\underbrace{x + x'}_{x''}, 0, \underbrace{z + z'}_{z''}) \in \mathbb{H} \text{ (}\mathbb{H} \text{ is closed under addition)}$$

3. Multiplying a vector in \mathbb{H} by a scalar produces another vector in \mathbb{H} (\mathbb{H} is closed under scalar multiplication).

Since properties 1, 2, and 3 hold, \mathbb{H} is a subspace of \mathbb{R}^3 .

Note : Vectors $(x, 0, z)$ in \mathbb{H} look and act like the points (a, b) in \mathbb{R}^2 .

Example 5. Let $S = \{(x, x + 1) : x \in \mathbb{R}\}$. Does S a subspace of \mathbb{R}^2 ?

Solution : S is not a subspace of \mathbb{R}^2 because $0_{\mathbb{R}^2} = (0, 0) \notin S$

Example 6. Let $\mathbb{V} = \{f : [0, 1] \rightarrow \mathbb{R}\}$ subspace of the vector space $\mathcal{F}(\mathbb{R})$ and

$$\mathbb{U} = \{f \in \mathbb{V} \mid f(1/3) = f(1/2) = 0\}, \quad \mathbb{W} = \{f \in \mathbb{V} \mid f(2/3) = 0\}$$

Prove that \mathbb{V} is subspace of \mathbb{V}

Solution : Let $f, g \in \mathbb{U}$ and $\alpha \in \mathbb{R}$, we have

1. $0(1/3) = 0(1/2) = 0$, thus zero function $0_{\mathbb{U}}$ in \mathbb{U}
2. $(f + g)(1/3) = \underbrace{f(1/3)}_{=0} + \underbrace{g(1/3)}_{=0} = 0$ and $(f + g)(1/2) = \underbrace{f(1/2)}_{=0} + \underbrace{g(1/2)}_{=0} = 0$, thus $f + g \in \mathbb{U}$
3. $(\alpha f)(1/3) = \alpha \underbrace{f(1/3)}_{=0} = 0$ and $(\alpha f)(1/2) = \alpha \underbrace{f(1/2)}_{=0} = 0$, thus $(\alpha f) \in \mathbb{U}$.

Therefore \mathbb{U} is subspace of \mathbb{V} .

The sum is

$$\mathbb{U} + \mathbb{W} = \{f + g \mid f \in \mathbb{U}, g \in \mathbb{W}\}$$

Exercise 3. Let $\mathbb{V} = \{f \in \mathcal{F}(\mathbb{R}) \mid \int_{-1}^1 f(x)dx = 0\}$. Is \mathbb{V} a real subspace ?

Solution 3. 1. The zero function $0(x) = 0$ is in \mathbb{V} since

$$\int_{-1}^1 0(x)dx = \int_{-1}^1 0dx = 0$$

2. Let $f, g \in \mathbb{V}$, use linearity property of the definite integral to calculate

$$\int_{-1}^1 (f(x) + g(x))dx = \int_{-1}^1 f(x)dx + \int_{-1}^1 g(x)dx = 0 + 0 = 0,$$

thus $f + g \in \mathbb{V}$

3. Let $f \in \mathbb{V}$, $\alpha \in \mathbb{R}$

$$\int_{-1}^1 \alpha f(x)dx = \alpha \int_{-1}^1 f(x)dx = \alpha 0 = 0$$

thus $\alpha f \in \mathbb{V}$. Therefore \mathbb{V} is real subspace

Theorem 1.1.2. *An arbitrary intersection of subspaces i.e. the intersection of any family of subspace of a vector space is a subspace.*

$$\text{If } (\mathbb{V}_i)_{i \in I} \text{ are vector spaces } \Rightarrow \mathbb{V} = \bigcap_{i \in I} \mathbb{V}_i \text{ is subspace}$$

Proof Let \mathbb{V} be a vector space over a field \mathbb{F} and $\mathbb{V}_i \subset \mathbb{V}$ be a subspaces of \mathbb{V} .

1. Since, $0_{\mathbb{V}} \in \mathbb{V}_i, \forall i \in I$, then $0_{\mathbb{V}} \in \bigcap_{i \in I} \mathbb{V}_i$. Hence, $\bigcap_{i \in I} \mathbb{V}_i \neq \Phi$
2. Let $x, y \in \bigcap_{i \in I} \mathbb{V}_i$ and $\alpha, \beta \in F$, then $x, y \in \mathbb{V}_i, \forall i \in I$ since every \mathbb{V}_i is a subspace of \mathbb{V} .

Consequently,

$$\alpha x + \beta y \in \mathbb{V}_i, \forall i \in I \Rightarrow \alpha x + \beta y \in \bigcap_{i \in I} \mathbb{V}_i$$

Hence, $\bigcap_{i \in I} \mathbb{V}_i$ is a subspace of \mathbb{V} .

Theorem 1.1.3. *The union of two subspace is a subspace if and only if one contains the other*

$$\text{If } \mathbb{V}_1, \mathbb{V}_2 \text{ are vector spaces, } \mathbb{V}_1 \cup \mathbb{V}_2 \text{ is a vector space } \Leftrightarrow \mathbb{V}_1 \subset \mathbb{V}_2 \text{ or } \mathbb{V}_2 \subset \mathbb{V}_1$$

Proof :

Let \mathbb{V} be a vector space over a field F and $\mathbb{V}_1, \mathbb{V}_2$ be two subspaces of \mathbb{V} .

Firstly let us suppose $\mathbb{V}_1 \subset \mathbb{V}_2$ or $\mathbb{V}_2 \subset \mathbb{V}_1$, then $\mathbb{V}_1 \cup \mathbb{V}_2 = \mathbb{V}_2$ or $\mathbb{V}_1 \cup \mathbb{V}_2 = \mathbb{V}_1$.

Therefore $\mathbb{V}_1 \cup \mathbb{V}_2$ is also a subspace of \mathbb{V} since $\mathbb{V}_1, \mathbb{V}_2$ are subspaces.

conversely, let $\mathbb{V}_1 \cup \mathbb{V}_2$ be a subspaces of \mathbb{V} , then we are to prove $\mathbb{V}_1 \subset \mathbb{V}_2$ or $\mathbb{V}_2 \subset \mathbb{V}_1$.

We shall prove it by contradiction.

Suppose that \mathbb{V}_1 is not a subspace of \mathbb{V}_2 and \mathbb{V}_2 is not a subspace of \mathbb{V}_1

Since

$$\mathbb{V}_1 \not\subset \mathbb{V}_2 \Rightarrow \exists x \in \mathbb{V}_1 \text{ and } x \notin \mathbb{V}_2, \quad (1.1)$$

$$\mathbb{V}_2 \not\subset \mathbb{V}_1 \Rightarrow \exists y \in \mathbb{V}_2 \text{ and } y \notin \mathbb{V}_1, \quad (1.2)$$

Now from (1) and (2), we get

$$x \in \mathbb{V}_1 \Rightarrow x \in \mathbb{V}_1 \cup \mathbb{V}_2 \quad \text{and} \quad y \in \mathbb{V}_2 \Rightarrow y \in \mathbb{V}_1 \cup \mathbb{V}_2.$$

Again since $\mathbb{V}_1 \cup \mathbb{V}_2$ is a subspace and so we have

$$x, y \in \mathbb{V}_1 \cup \mathbb{V}_2 \Rightarrow x + y \in \mathbb{V}_1 \cup \mathbb{V}_2 \Rightarrow x + y \in \mathbb{V}_1 \text{ or } x + y \in \mathbb{V}_2.$$

If $x + y \in \mathbb{V}_1$, then $(x + y) - x = y \in \mathbb{V}_1$ since \mathbb{V}_1 is a subspace and $y \in \mathbb{V}_1$. But from (2), we see that $y \notin \mathbb{V}_1$, which is contradiction .

Hence either \mathbb{V}_1 is a subspace of \mathbb{V}_2 or \mathbb{V}_2 is a subspace of \mathbb{V}_1 .

1.1.2 Product Space of vector spaces :

Lemma 1.1.1. *Product Space of vector spaces is vector space*

If \mathbb{V}_1 and \mathbb{V}_2 are both vector spaces over a field \mathbb{F} then,

$$\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 = \{(x, y) \mid x \in \mathbb{V}_1, y \in \mathbb{V}_2\}$$

Forms a vector space over \mathbb{F} subject to : $\forall (x, y), (x', y') \in \mathbb{V}_1 \times \mathbb{V}_2, \forall \alpha \in \mathbb{F}$

$$(x, y) \oplus (x', y') := \left(\underbrace{x + x'}_{\text{addition in } \mathbb{V}_1}, \underbrace{y + y'}_{\text{addition in } \mathbb{V}_2} \right), \quad \alpha \odot (x, y) = \left(\underbrace{\alpha x}_{\text{multiplication in } \mathbb{V}_1}, \underbrace{\alpha y}_{\text{multiplication in } \mathbb{V}_2} \right)$$

Example 7. \mathbb{R} is real vector space, then $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is also real vector space

1.1.3 Linear combinations of vectors

Definition 1.1.3. *In a vector space \mathbb{V} over a field \mathbb{F} we are free to form \mathbb{F} -linear combinations of vectors, we say v is a **linear combination** of $v_1, v_2, \dots, v_p \in \mathbb{V}$ if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_p \in \mathbb{F}$, such that :*

$$v = \sum_1^p \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p$$

Example 8. Let $x_1 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$, $x_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$. Is $u = \begin{pmatrix} -7 \\ -15 \\ 6 \end{pmatrix}$ a linear combination of x_1, x_2, x_3 ?

Solution :

$$u \text{ a linear combination of } x_1, x_2, x_3 \Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{R} : u = \alpha x_1 + \beta x_2 + \gamma x_3$$

So,

$$\begin{aligned} u = \alpha x_1 + \beta x_2 + \gamma x_3 &\Leftrightarrow \begin{pmatrix} -7 \\ -15 \\ 6 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} -7 &= 3\beta + 2\gamma \\ -15 &= 2\alpha + 5\beta - 4\gamma \\ 6 &= 4\alpha + \beta + \gamma \end{cases} \end{aligned}$$

We get $\alpha = 2, \beta = -3, \gamma = 1$, therefore u is linear combination of x_1, x_2, x_3 .

Exercise 4. Let $x_1 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$, $x_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$. Is $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of x_1, x_2, x_3 ?

Solution 4.

u a linear combination of $x_1, x_2, x_3 \Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{R} : u = \alpha x_1 + \beta x_2 + \gamma x_3$

So,

$$\begin{aligned} u = \alpha x_1 + \beta x_2 + \gamma x_3 &\Leftrightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} 1 = 3\beta + 3\gamma \\ 2 = 2\alpha + 5\beta + 7\gamma \\ 3 = 4\alpha + \beta + 5\gamma \end{cases} \quad \text{No solution} \end{aligned}$$

Therefore, u is not a linear combination of x_1, x_2, x_3 .

Exercise 5. Let $x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $x_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $x_4 = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$. Is $u = \begin{pmatrix} 13 \\ 8 \\ 13 \end{pmatrix}$ a linear combination of x_1, x_2, x_3, x_4 ?

Solution 5. yes u is a linear combination of x_1, x_2, x_3, x_4 , we get

$$\begin{pmatrix} 13 \\ 8 \\ 13 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} + 5 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

1.1.4 Linearly independent vectors, Linearly dependent vectors

Definition 1.1.4. Let \mathbb{V} be a vector space over a field F . A finite set $\{u_1, u_2, \dots, u_n\}$ of vectors of \mathbb{V} is said to be **linearly independent** if every relation of the form

$$\forall \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}, \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0_{\mathbb{V}} \Rightarrow \alpha_i = 0_{\mathbb{F}} \quad \text{for each } 1 \leq i \leq n$$

Definition 1.1.5. Let \mathbb{V} be a vector space over a field F . A finite set $\{u_1, u_2, \dots, u_n\}$ of vectors of \mathbb{V} is said to be **linearly dependent** if there exist scalar $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ not of them $0_{\mathbb{F}}$ (some of them may be zero) such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0_{\mathbb{V}}$$

An infinite set of vectors of \mathbb{V} is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Example 9. Find whether the set of vectors : $x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$, $x_3 = \begin{pmatrix} 3 \\ -4 \\ 7 \end{pmatrix}$ is linearly independent or dependent.

Solution : Let α, β, γ be three scalars such that

$$\alpha x_1 + \beta x_2 + \gamma x_3 = 0_{\mathbb{R}^3}$$

Thus

$$\begin{aligned}
 (\alpha x_1 + \beta x_2 + \gamma x_3 = 0_{\mathbb{R}^3}) &\Leftrightarrow \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
 &\Leftrightarrow \begin{cases} \alpha + 3\beta + 3\gamma = 0 \\ 2\alpha + \beta - 4\gamma = 0 \\ \alpha + 5\beta + 7\gamma = 0 \end{cases}
 \end{aligned}$$

We get, $\alpha = -3$, $\beta = 2$, $\gamma = -1$. Hence the set of vectors $\{x_1, x_2, x_3\}$ are linearly dependent.

Example 10. Show that the set $\{1, x, 1 + x + x^2\}$ is linearly independent set of vectors in the vector space of all polynomial over the real number field.

Solution : Let α, β, γ be three scalars(real numbers) such that

$$\alpha(1) + \beta(x) + \gamma(1 + x + x^2) = 0_{\mathbb{R}} = 0$$

We have

$$\begin{aligned}
 \alpha(1) + \beta(x) + \gamma(1 + x + x^2) = 0 &\Rightarrow (\alpha + \gamma) + (\beta + \gamma)x + \gamma x^2 = 0, \\
 &\Rightarrow \begin{cases} \alpha + \gamma = 0 \\ \beta + \gamma = 0 \\ \gamma = 0 \end{cases} \\
 &\Rightarrow \alpha = \beta = \gamma = 0.
 \end{aligned}$$

Therefore the vectors $1, x, 1 + x + x^2$ are linearly independent over a field of real numbers.

Exercise 6. Are the vectors $x_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ -2 \\ -4 \\ 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 4 \\ -2 \\ -5 \\ 2 \end{pmatrix}$, $x_4 = \begin{pmatrix} 4 \\ 2 \\ 1 \\ 6 \end{pmatrix}$ linearly independent or dependent.

Solution 6. Let $\alpha, \beta, \gamma, \delta$ be three scalars(real numbers) such that

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0_{\mathbb{R}^4}$$

We have

$$\begin{aligned}
 \alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0_{\mathbb{R}^4} &\Rightarrow \alpha \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -2 \\ -4 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ -2 \\ -5 \\ 2 \end{pmatrix} + \delta \begin{pmatrix} 4 \\ 2 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
 &\Rightarrow \begin{cases} 2\alpha + 2\beta + 4\gamma + 4\delta = 0 \\ 2\alpha - 2\beta - 2\gamma + 2\delta = 0 \\ 2\alpha - 4\beta - 5\gamma + \delta = 0 \\ 4\alpha + 2\gamma + 6\delta = 0 \end{cases} \\
 &\Rightarrow \begin{cases} 2\alpha + 2\beta + 4\gamma + 4\delta = 0 \\ 0\alpha - 4\beta - 6\gamma - 2\delta = 0 \\ 0\alpha + 0\beta + 0\gamma + 0\delta = 0 \\ 0\alpha + 0\beta + 0\gamma + 0\delta = 0 \end{cases}
 \end{aligned}$$

The system of equations will have $4 - 2 = 2$, non-zero solutions and hence the set of vectors are linearly dependent. Hence given vectors are not linearly independent.

1.1.5 Spanning sets and subspaces

Definition 1.1.6. Let u_1, u_2, \dots, u_p be p vectors in \mathbb{V} . Define the set \mathbb{W} as the collection of all linear combination of u_1, u_2, \dots, u_p

$$\begin{aligned} \mathbb{W} &= \{ \text{the collection of all linear combination of } u_1, u_2, \dots, u_p \}, \\ &= \left\{ u \in \mathbb{V} \mid u = \sum_1^p \alpha_i u_i, \alpha_i \in \mathbb{F}, i = 1, 2, \dots, p \right\} \subseteq \mathbb{V} \end{aligned}$$

Theorem 1.1.4. $\mathbb{W} = \{u \in \mathbb{V} \mid u = \sum_1^p \alpha_i u_i, \alpha_i \in \mathbb{F}, i = 1, 2, \dots, p\}$ is subspace of \mathbb{V} over a field \mathbb{F} .

Proof : In order to verify this, check properties of definition of a subspace.

1. $0_{\mathbb{V}} \in \mathbb{W}$ since if $(\alpha_i = 0_{\mathbb{F}}, \forall i = 1, 2, \dots, p)$, we have $u = 0_{\mathbb{F}}u_1 + 0_{\mathbb{F}}u_2 + \dots + 0_{\mathbb{F}}u_p = 0_{\mathbb{V}} \in \mathbb{W}$
2. \mathbb{W} is closed under addition since

$$\forall \alpha_i, \beta_i, \in \mathbb{F}, i = 1, 2, \dots, p \in \mathbb{F}, \sum_1^p \alpha_i u_i + \sum_1^p \beta_i u_i = \left(\sum_1^p \underbrace{(\alpha_i + \beta_i)}_{\gamma_i \in \mathbb{F}} u_i \right) \in \mathbb{W}$$

3. \mathbb{W} is closed under scalar multiplication since

$$\forall \beta, \alpha_i \in \mathbb{F}, i = 1, 2, \dots, p \in \mathbb{F}, \beta \left(\sum_1^p \alpha_i u_i \right) = \left(\sum_1^p \underbrace{\beta \alpha_i}_{\beta_i \in \mathbb{F}} u_i \right) \in \mathbb{W}$$

Since properties of definition of a subspace hold. $\text{span} \{u_1, \dots, u_p\}$ is a subspace of \mathbb{W} . \mathbb{W} is called the span of u_1, u_2, \dots, u_p and denoted $\mathbb{W} = \text{span} \{u_1, u_2, \dots, u_p\}$

Definition 1.1.7. Let $S = \{u_1, u_2, \dots, u_p\}$ be a finite set of vectors in vector space \mathbb{V} over a field \mathbb{F} then Span (or linear closure) is defined to be the set of all \mathbb{F} -linear combinations of S .

$$\text{span}(S) = \langle S \rangle = \{ \alpha u_1 + \alpha u_2 + \dots + \alpha_p u_p \mid \alpha_1, \alpha_2, \dots, \alpha_p \in \mathbb{F} \}$$

If $\mathbb{V} = \text{span}(S)$, then we say that S is a **generating set or spanning set** for \mathbb{V} . We also say S spans \mathbb{V} .

Finally, $\text{span}(\emptyset) = \{0_{\mathbb{V}}\}$ (convention)

Remark 1.1.1. $\text{span} \{u_1, \dots, u_p\}$ is a subspace of \mathbb{V} . It is the **smallest subspace of \mathbb{V} which contains S** . In this sense that if \mathbb{H} is any subspace which includes u_1, u_2, \dots, u_p , then $\text{span}S \subset \mathbb{H}$.

Results of definition :

1. $\text{span}(S)$ is always a subspace of \mathbb{V} (the smallest subspace of \mathbb{V})
2. If a vector $u \in \text{span}(S)$, then u is a linear combination of S
3. If u is a linear combination of S , then $u \in \text{span}(S)$.

Example 11. Find the subspace of \mathbb{V} that is spanned by

$$S = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Solution : Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, then

$$\begin{aligned} v \in \text{span}(S) &\Leftrightarrow \exists \alpha, \beta \in \mathbb{R} : v = \alpha u_1 + \beta u_2, \\ &\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \\ &\Leftrightarrow \begin{cases} x = \alpha - \beta \\ y = \alpha \\ z = 3\alpha + 2\beta \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = y \\ x = y - \beta \\ z = 3y + 2\beta \end{cases} \\ &\Leftrightarrow 2x - y + z = 0 \end{aligned}$$

Hence the vector space \mathbb{V} is the set of vectors defined by

$$V = \text{span}(S) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \in \mathbb{R} \mid 2x - y + z = 0 \right\}$$

V is a plane through the origin

Example 12. Find the subspace of \mathbb{R}^4 that is spanned by :

$$S = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}, u_4 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 4 \end{pmatrix} \right\}$$

Solution : Let $v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4$, then

$$\begin{aligned} v \in \text{span}(S) &\Leftrightarrow \exists \alpha, \beta, \gamma, \delta \in \mathbb{R} : v = \alpha u_1 + \beta u_2 + \gamma u_3 + \delta u_4, \\ &\Leftrightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + \delta \begin{pmatrix} 3 \\ 3 \\ 3 \\ 4 \end{pmatrix}, \alpha, \beta, \gamma, \delta \in \mathbb{R}, \\ &\Leftrightarrow \begin{cases} x = \alpha + 2\beta + 3\delta \\ y = \alpha + 2\beta + 3\delta \\ z = \alpha + 2\beta + 3\delta \\ t = \alpha + 3\gamma + 4\delta \end{cases} \Leftrightarrow x = y = z, \text{ and } t \text{ free variable} \end{aligned}$$

Hence the vector space \mathbb{V} is the set of vectors defined by

$$V = \text{span}(S) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mid x, y, z, t \in \mathbb{R} \mid x = y = z \right\} = \left\{ \begin{pmatrix} x \\ x \\ x \\ t \end{pmatrix} \mid x, t \in \mathbb{R} \right\}$$

Exercise 7. Find the subspace of \mathbb{V} that is spanned by

$$S = \{u_1 = 3 + x, u_2 = 2 + x^2, u_3 = x^3\}.$$

Solution 7. Let $P \in P_n = \mathbb{R}_n[X]$, we have

$$\begin{aligned} P \in \text{span}(S) &\Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{R}, P(x) = \alpha u_1 + \beta u_2 + \gamma u_3, \\ &\Leftrightarrow P(x) = \alpha(3 + x) + \beta(2 + x^2) + \gamma x^3, \alpha, \beta, \gamma \in \mathbb{R}, \\ &\Leftrightarrow P(x) = \gamma x^3 + \beta x^2 + \alpha x + 3\alpha + 2\beta, \alpha, \beta, \gamma \in \mathbb{R} \end{aligned}$$

Hence the vector space \mathbb{V} is the set of vectors defined by

$$\mathbb{V} = \text{span}(S) = \{P(x) \in \mathbb{R}_3[X] \mid P(x) = \gamma x^3 + \beta x^2 + \alpha x + 3\alpha + 2\beta, \alpha, \beta, \gamma \in \mathbb{R}\}$$

Important note : $\mathbb{V} \neq \mathbb{R}_3[X]$. Not all polynomials of degree 3 belong to the set \mathbb{V} .

Let's check. If $P(x) = x^3 + x^2 + 1 \in \mathbb{R}_3[X]$

$$\begin{aligned} P(x) \in \text{span}(S) &\Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{R}, P(x) = \gamma x^3 + \beta x^2 + \alpha x + 3\alpha + 2\beta, \\ &\Leftrightarrow x^3 + x^2 + 1 = \gamma x^3 + \beta x^2 + \alpha x + 3\alpha + 2\beta, \\ &\Leftrightarrow \begin{cases} \gamma &= 1 \\ \beta &= 1 \\ \alpha &= 0 \\ 3\alpha + 2\beta &= 1 \end{cases} \text{ No solution} \end{aligned}$$

Thus $P(x) \notin \text{span}(S)$

Exercise 8. Let $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Does $\{u_1, u_2\}$ span \mathbb{R}^2 ?

Solution 8. Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Can we write

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} ??$$

So, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix} \Rightarrow \begin{cases} a = x_1 + x_2 \\ b = x_1 - 2x_2 \end{cases}$$

we get

$$\begin{cases} x_1 = \frac{2}{3}a + \frac{1}{3}b \\ x_2 = \frac{1}{3}a - \frac{1}{3}b \end{cases}$$

Therefore, $\{u_1, u_2\}$ span \mathbb{R}^2

e.g Let $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, we have

$$\begin{cases} x_1 = \frac{2}{3}4 + \frac{1}{3}(-2) = 2 \\ x_2 = \frac{1}{3}4 - \frac{1}{3}(-2) = 2 \end{cases}$$

Then,

$$\begin{pmatrix} 4 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Exercise 9. Let $u_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. Does $\{u_1, u_2\}$ span \mathbb{R}^2 ?

Solution 9. Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Can we write

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 4 \end{pmatrix} ??$$

So, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ -2x_1 + 4x_2 \end{pmatrix} \Rightarrow \begin{cases} a = x_1 - 2x_2 \\ b = -2x_1 + 4x_2 \end{cases} \text{ No solution}$$

Therefore, the vectors u_1, u_2 do not span \mathbb{R}^2 .

◇ **Determining if vector belongs to span**

Let \mathbb{V} be a vector space over a field \mathbb{F} , $\mathbb{S} = \text{span}\{u_1, u_2, \dots, u_p\}$ and $u \in \mathbb{V}$.

$$\begin{aligned} u \in \text{span}(S) &\Leftrightarrow u, u_1, u_2, \dots, u_p \text{ are linearly dependent} \\ &\Leftrightarrow u \text{ is linear combination of } u_1, u_2, \dots, u_p \\ &\Leftrightarrow \exists \alpha_i \in \mathbb{R}, i = 1, 2, \dots, p, u = \sum_1^p \alpha_i u_i \end{aligned}$$

Example 13. Let $S = \left\{ u_1 = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \right\}$. Is the vector $u = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ in $\text{span}(S)$?

Solution :

$$\begin{aligned} u \in \text{span}(S) &\Leftrightarrow u, u_1, u_2 \text{ are linearly dependent} \\ &\Leftrightarrow u \text{ is linear combination of } u_1, u_2 \end{aligned}$$

Can u be written as $u = \alpha u_1 + \beta u_2$ where $\alpha, \beta \in \mathbb{R}$.

So are there $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Is there a solution to

$$\begin{cases} -3\alpha + 5\beta = 1 \\ \alpha - 2\beta = -1 \\ -\alpha + \beta = -1 \end{cases}$$

The system has one solution (single solution) $\alpha = 3, \beta = 2$ therefore $u \in \text{span}(S)$.

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 3 \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Exercise 10.

Let $S = \left\{ u_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$. Is the vector $u = \begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ in $\text{span}(S)$?

Solution 10.

$$\begin{aligned}
u \in \text{span}(S) &\Leftrightarrow u, u_1, u_2, u_3 \text{ are linearly dependent} \\
&\Leftrightarrow u \text{ is linear combination of } u_1, u_2, \text{ and } u_3 \\
&\Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{R}, u = \alpha u_1 + \beta u_2 + \gamma u_3, \\
&\Leftrightarrow \begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \alpha, \beta, \gamma \in \mathbb{R}, \\
&\Leftrightarrow \begin{cases} 2 &= 2\alpha + 3\beta - \gamma \\ 3 &= \alpha - \beta \\ -7 &= 5\beta + 2\gamma \\ 3 &= 3\alpha + 2\beta + \gamma \end{cases} \Leftrightarrow \alpha = 2, \beta = -1, \gamma = -1 \text{ One solution}
\end{aligned}$$

The system has one solution, therefore $u \in \text{span}(S)$.

Exercise 11.

Prove that

$$\begin{pmatrix} 2 \\ 14 \\ -3 \\ -15 \\ -2 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \\ -5 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

Solution 11. We get

$$\begin{pmatrix} 2 \\ 14 \\ -3 \\ -15 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 7 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ -3 \\ -5 \\ -6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -3 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

1.1.6 Basis and Dimension

Definition 1.1.8. Any subset \mathcal{B} of a vector space \mathbb{V} over a field \mathbb{F} is called **basis** of \mathbb{V} if :

1. \mathcal{B} is linearly independent
2. \mathcal{B} generates \mathbb{V} i.e. $V = \langle \mathcal{B} \rangle = \text{span} \{ \mathcal{B} \}$ (\mathcal{B} is span of \mathbb{V})

A basis of a vectors space \mathbb{V} is a linearly independent set which spans \mathbb{V} .
In other words \mathcal{B} is a basis of \mathbb{V} if each vector in \mathbb{V} is uniquely expressed as a linear combination of the basis vector

Example 14.

1. $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is the standard basis of the space vector \mathbb{R}^2
2. $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is the standard basis of the space vector \mathbb{R}^3

3. In general $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$ is the standard basis of the space vector \mathbb{R}^n .

4. P_n the set of all polynomials of degree at most n . Members of P_n have the form

$$P_n = \{p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n, \deg(p(x)) \leq n\}$$

where a_0, a_1, \dots, a_n are real numbers and x is a real variable. $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$ is standard basis of P_n

Exercise 12. Show that the set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ form a basis for \mathbb{R}^3 .

Solution 12. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

1. **Linearly independent?**

$$\begin{aligned} \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} &\Leftrightarrow \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \beta \\ \beta \\ 0 \end{pmatrix} + \begin{pmatrix} \gamma \\ \gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ \beta + \gamma = 0 \\ \gamma = 0 \end{cases} \\ &\Leftrightarrow \alpha = \beta = \gamma = 0. \end{aligned}$$

Hence vectors are linearly independent

2. **\mathcal{B} generates \mathbb{R}^3 ?**

\mathcal{B} generates \mathbb{R}^3 if $\forall u \in \mathbb{R}^3, \exists \alpha, \beta, \gamma \in \mathbb{R}$ such that

$$u = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an arbitrary vector in \mathbb{R}^3 ,

$$\begin{aligned} u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \beta \\ \beta \\ 0 \end{pmatrix} + \begin{pmatrix} \gamma \\ \gamma \\ \gamma \end{pmatrix} \\ &\Leftrightarrow \begin{cases} x = \alpha + \beta + \gamma \\ y = \beta + \gamma \\ z = \gamma \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = x - y \\ \beta = y - z \\ \gamma = z \end{cases} \end{aligned}$$

Therefore \mathcal{B} generates \mathbb{R}^3 . Hence \mathcal{B} is a basis of \mathbb{R}^3

Exercise 13. Consider $\mathcal{B} = \{(t+1)^2, t+1, 1\}$. Calculate the coordinate vector of $u = t^2$ with respect to basis \mathcal{B} .

Solution 13. Let $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ be the coordinate of the vector u with respect to basis \mathcal{B} , so

$$\begin{aligned} t^2 = u &= \alpha(t+1)^2 + \beta(t+1) + \gamma \\ &= \alpha t^2 + (2\alpha + \beta)t + \alpha + \beta + \gamma \end{aligned}$$

Thus

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ 2\alpha + \beta = 0 \\ \alpha = 1 \end{cases} \implies \begin{cases} \alpha = 1 \\ \beta = -2 \\ \gamma = 1 \end{cases} \implies [u]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Definition 1.1.9. The number of element in the finite basis of a vector space \mathbb{V} is called the dimension of the vector space. It is denoted by $\dim(V)$ and if $\dim(V) = n$, then \mathbb{V} is called n -dimensional vector space.

1. **Finite Dimensional Vector Space (FDVS)** : A vector space \mathbb{V} is said to be finite dimensional vector space if it has a finite basis.
2. **Infinite Dimensional Vector Space (IDVS)** : A vector space \mathbb{V} is said to be infinite dimensional vector space if it has a infinite basis.

Example 15.

1. \mathbb{C} over a field \mathbb{R} is a vector space of dimension of two (2) as its basis is $\mathcal{B} = \{1, i\}$ as $\forall z \in \mathbb{C} : \exists a, b \in \mathbb{R}; z = a + bi$
2. \mathbb{C} over a field \mathbb{C} is a vector space of dimension of one (1) as its basis is $\mathcal{B} = \{1\}$
3. In general : Dimension of field F over itself F is always one as basis $\mathcal{B} = \{1\}$

4. Dimension of \mathbb{R}^2 over \mathbb{R} is two (2) as basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ as

$$\forall u = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : u = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

5. In general : Dimension of vector space F^n over F is n -dimensional space as its

$$\text{standard basis will be } \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

6. \mathbb{C} over \mathbb{Q} and \mathbb{R} over \mathbb{Q} are of infinite dimensional vector spaces.

7. Dimension of P_n over \mathbb{R} is $n + 1$ as basis $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$

Convention :

$$\dim(\{0_{\mathbb{V}}\}) = 0$$

Example 16. Show that the following set \mathcal{B} of \mathbb{R}^3 form a basis for \mathbb{R}^3 :

$$\mathcal{B} = \left\{ u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

Express each of the standard basis vectors $\{e_1, e_2, e_3\}$ as linear combination of the above basis vectors.

Solution :

1. We Know that the set of the vectors $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ form a

basis of \mathbb{R}^3 . Therefore $\dim(\mathbb{R}^3) = 3$. Further, we know that any set of n linearly independent vectors of n -dimensional vector space \mathbb{V} form a basis for \mathbb{V} , so that in order to show that \mathcal{B} forms basis for \mathbb{R}^3 it is just sufficient to show that the set \mathcal{B} is linearly independent. Now since.

Let $\alpha, \beta, \gamma \in \mathbb{R}$, and

$$\begin{aligned} \alpha u_1 + \beta u_2 + \gamma u_3 = 0_{\mathbb{R}^3} &\Leftrightarrow \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} \alpha + 2\beta + \gamma = 0 \\ 2\alpha + \beta - \gamma = 0 \\ \alpha + 2\beta = 0 \end{cases} \\ &\Leftrightarrow \alpha = \beta = \gamma = 0 \end{aligned}$$

Therefore, the set \mathcal{B} is linear independent. Hence \mathcal{B} forms a basis of \mathbb{R}^3 .

2.

$$\begin{aligned}
 e_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \\
 &= \begin{pmatrix} a + 2b + c \\ 2a + b - c \\ a + b + 2c \end{pmatrix}.
 \end{aligned}$$

So, we have

$$\begin{cases} a + 2b + c = 1 \\ 2a + b - c = 0 \\ a + b + 2c = 0 \end{cases} \Leftrightarrow \begin{cases} a = 2/9 \\ b = 5/9 \\ c = 1/6 \end{cases}$$

Thus showing that e_1 can be expressed a linear combination of elements of \mathcal{B} .

Similarly

$$\begin{aligned}
 e_2 &= \frac{4}{9}u_1 - \frac{1}{9}u_2 - \frac{2}{9}u_3, \\
 e_3 &= \frac{1}{3}u_1 - \frac{1}{3}u_2 + \frac{1}{3}u_3.
 \end{aligned}$$

Exercise 14. Find the co-ordinate vector of $u = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$. relative to the basis

$$\mathcal{B} = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Solution 14. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$u = \alpha u_1 + \beta u_2 + \gamma u_3$$

We get

$$\alpha = 3, \beta = -2, \gamma = -5$$

Thus the coordinate vector of u relative to the given basis is $[u]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}_{\mathcal{B}}$

1.1.7 Spanning and dimension

Question :Let \mathbb{V} be a vector space and $S = \{u_1, u_2, \dots, u_p\} \subset V$. Does $\text{span}(S) = V$.**Solution :** Let $\dim(V) = n$

1. If $p < n$ then, $\text{span}(S) \neq V$
2. If $p = n$ then, $\text{span}(S) = V \Leftrightarrow S$ linearly independent
3. $p > n$ then, we must solve the systems $v = \sum_1^p \alpha_i u_i$, ($v \in V$)

Example 17. Let $S \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$. Does $\text{span}(S) = \mathbb{R}^3$?

Solution : we have $\dim(\text{span}(S)) = 2 < \dim(\mathbb{R}^3) = 3$ thus, $\text{span}(S) \neq \mathbb{R}^3$,

Example 18. Let $S \{3 + x, x^2, x^3\}$. Does $\text{span}(S) = P_3 = \mathbb{R}_3[X]$?

Solution : First we see that $3 + x, x^2, x^3 \in P_3$.
we have

$$\dim(\text{span}(S)) = 3 < \dim(P_3) = 3 + 1 = 4$$

thus, $\text{span}(S) \neq P_3$,

Example 19. Let $S \{u_1 = 2 - x + 3x^2, u_2 = 4 + x + 2x^2, u_3 = 8 - x + 8x^2\}$. Does $\text{span}(S) = P_2$?

Solution :

First we see that $2 - x + 3x^2, 4 + x + 2x^2, 8 - x + 8x^2 \in P_2$, we have

$$\dim(\text{span}(S)) = 3 = \dim(P_2) = 2 + 1 = 3$$

So,

$\text{span}(S) = P_2 \Leftrightarrow u_1, u_2, u_3$ Linearly independent

$$\Leftrightarrow (\alpha u_1 + \beta u_2 + \gamma u_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0, \alpha, \beta, \gamma \in \mathbb{R})$$

$$\Leftrightarrow \alpha(2 - x + 3x^2) + \beta(4 + x + 2x^2) + \gamma(8 - x + 8x^2) = 0, \alpha, \beta, \gamma \in \mathbb{R}$$

$$\Leftrightarrow \begin{cases} 2\alpha + 4\beta + 8\gamma = 0 \\ -\alpha + \beta - \gamma = 0 \\ 3\alpha + 2\beta + 8\gamma = 0 \end{cases} \Leftrightarrow \alpha = -2, \beta = -1, \gamma = 1.$$

More than just the trivial solution, therefore u_1, u_2, u_3 are not linearly independent (linearly dependent). Thus $\text{span}(S) \neq P_2$.

Example 20. Let $S \{u_1 = 1 - x + 2x^2, u_2 = 3 + x, u_3 = 5 - x + 4x^2, u_4 = -2 - 2x + 2x^2\}$. Does $\text{span}(S) = P_2$?

Solution :

First we see that $u_1, u_2, u_3, u_4 \in P_2$, we have

$$\dim(\text{span}(S)) = 4 > \dim(P_2) = 2 + 1 = 3$$

So,

$$\text{span}(S) = P_2 \Leftrightarrow \forall P \in P_2, \exists \alpha, \beta, \gamma, \delta \in \mathbb{R}, P(x) = \alpha u_1 + \beta u_2 + \gamma u_3 + \delta u_4$$

Let $P(x) = a + bx + cx^2 \in P_2$, we get

$$P(x) \in \text{span}(S) \Leftrightarrow \exists \alpha, \beta, \gamma, \delta \in \mathbb{R}, P(x) = \alpha u_1 + \beta u_2 + \gamma u_3 + \delta u_4$$

$$\Leftrightarrow a + bx + cx^2 = \alpha(1 - x + 2x^2) + \beta(3 + x) + \gamma(5 - x + 4x^2) + \delta(-2 - 2x + 2x^2),$$

$$\Leftrightarrow \begin{cases} \alpha + 3\beta + 5\gamma - 2\delta = a \\ -\alpha + \beta - \gamma - 2\delta = b \\ 2\alpha + 4\gamma + 2\delta = c \end{cases} \Leftrightarrow a - 3b - 2c = 0.$$

Thus $\text{span}(S)$ is the vector space defined by

$$\text{span}(S) = \{P(x) = a + bx + cx^2 \mid a - 3b - 2c = 0\} \neq P_2$$

1.1.8 Sum and Direct sums of subspaces

Definition 1.1.10. Let \mathbb{V} be vector space over field \mathbb{F} , \mathbb{U} and \mathbb{W} be two subspaces of \mathbb{V} . The **sum** of \mathbb{U} and \mathbb{W} is the following set :

$$\mathbb{U} + \mathbb{W} = \{u + w \mid u \in \mathbb{U}, w \in \mathbb{W}\}$$

- The sum $\mathbb{U} + \mathbb{W}$ is a subspace of \mathbb{V}
- If \mathbb{U} and \mathbb{W} are subspaces of a finite dimensional vector space \mathbb{V} , then

$$\dim(\mathbb{U} + \mathbb{W}) = \dim(\mathbb{U}) + \dim(\mathbb{W}) - \dim(\mathbb{U} \cap \mathbb{W})$$

In general : If \mathbb{V}_i , $i = 1, 2, \dots, p$ are subspaces of \mathbb{V} , the sum of \mathbb{V}_i $i = 1, 2, \dots, p$ is the subspace

$$\mathbb{V}_1 + \mathbb{V}_2 + \dots + \mathbb{V}_p = \{v_1 + v_2 + \dots + v_p \mid v_i \in \mathbb{V}_i, i = 1, 2, \dots, p\}$$

Example 21. In \mathbb{R}^3 , let $\mathbb{V} = \left\{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}, t \in \mathbb{R} \right\}$ and $\mathbb{W} = \left\{ \begin{pmatrix} t \\ 2t \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}$,

\mathbb{V} and \mathbb{W} are two subspaces of \mathbb{R}^3 and we have

$$\mathbb{V} + \mathbb{W} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} \beta \\ 2\beta \\ 0 \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\} = \left\{ \alpha \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{u_1} + \beta \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{u_2}, \alpha, \beta \in \mathbb{R} \right\}$$

So, $\mathbb{V} + \mathbb{W} = \text{span}\{u_1, u_2\}$, the vectors u_1, u_2 are linearly independent, thus $\{u_1, u_2\}$ forms a basis of $\mathbb{V} + \mathbb{W}$.

Geometrically :

\mathbb{V} and \mathbb{W} are two lines through the origin and $\mathbb{V} + \mathbb{W}$ is a plane and passing through the origin and generated by the two vectors u_1, u_2 .

Example 22. Let

$$\mathbb{V} = \{(x, 0), x \in \mathbb{R}\} \text{ and } \mathbb{W} = \{(0, y), y \in \mathbb{R}\}$$

It is clear that

$$\mathbb{V} + \mathbb{W} = \{(x, y), x, y \in \mathbb{R}\} = \mathbb{R}^2$$

Definition 1.1.11. The sum $\mathbb{U} + \mathbb{W}$ is called **direct**, if for each $x \in \mathbb{U} + \mathbb{W}$ there exist **uniquely determined** $u \in \mathbb{U}$ and $w \in \mathbb{W}$, such that $x = u + w$.

Notation of a direct sum : $\mathbb{U} \oplus \mathbb{W}$

$$\mathbb{U} \oplus \mathbb{W} = \{x \in \mathbb{V} \mid \exists! u \in \mathbb{U}, \exists! w \in \mathbb{W}, x = u + w\}$$

Example 23. Let \mathbb{V} and \mathbb{W} be two subspaces of \mathbb{R}^3 such that

$$\mathbb{V} = \left\{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}, c \in \mathbb{R} \right\}, \quad \mathbb{W} = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, a, b \in \mathbb{R} \right\}$$

Prove that $\mathbb{R}^3 = \mathbb{V} \oplus \mathbb{W}$.

Solution : any vector in \mathbb{R}^3 can be written as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in \mathbb{V} + \mathbb{W} \text{ in a unique way}$$

Thus, $\mathbb{R}^3 = \mathbb{V} \oplus \mathbb{W}$

Example 24. Let \mathbb{V} and \mathbb{W} be two subspaces of \mathbb{R}^3 such that

$$\mathbb{V} = \left\{ \begin{pmatrix} 0 \\ c \\ d \end{pmatrix} : c, d \in \mathbb{R} \right\}, \quad \mathbb{W} = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

We have,

$$\mathbb{V} + \mathbb{W} = \left\{ \begin{pmatrix} a \\ b + c \\ d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

There is no longer a unique way to write elements

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Thus, $\mathbb{R}^3 \neq \mathbb{V} \oplus \mathbb{W}$

Proposition 1.1.1. Let \mathbb{U}, \mathbb{W} be subspaces of \mathbb{V} . The following statements are equivalent :

1. The sum of \mathbb{U} and \mathbb{W} is direct $\mathbb{U} \oplus \mathbb{W}$

2.

$$u_1 + w_1 = u_2 + w_2 \Rightarrow u_1 = u_2 \text{ and } w_1 = w_2, \quad (u_i \in \mathbb{U}, w_i \in \mathbb{W}, i = 1, 2.)$$

3.

$$\forall (u, w) \in \mathbb{U} \times \mathbb{W}, u + w = 0_{\mathbb{V}} \Rightarrow u = w = 0_{\mathbb{V}}$$

4.

$$\mathbb{U} \cap \mathbb{W} = \{0_{\mathbb{V}}\}$$

5.

$$\dim(\mathbb{U} + \mathbb{W}) = \dim(\mathbb{U}) + \dim(\mathbb{W})$$

6. If \mathcal{B}_U is basis of subspace U and \mathcal{B}_W is basis of subspace W , then $\mathcal{B}_U \cup \mathcal{B}_W$ is basis of subspace $U + W$

Result :

The **necessary and sufficient conditions** for a vector space V to be **direct sum** of its subspaces U and W are

$$V = U + W \quad \text{and} \quad U \cap W = \{0_V\}$$

Example 25. Let

$$V = \{(x, 0), x \in \mathbb{R}\} \quad \text{and} \quad W = \{(0, y), y \in \mathbb{R}\}$$

It is clear that

$$V \oplus W = \{(x, y), x, y \in \mathbb{R}\} = \mathbb{R}^2$$

Example 26. Let V and W be two subspaces of \mathbb{R}^3 such that

$$V = \left\{ \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix}, y \in \mathbb{R} \right\}, \quad W = \left\{ \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, x, y \in \mathbb{R} \right\}$$

Prove that $\mathbb{R}^3 = V \oplus W$.

Solution : From the previous example (22), we get $V + W = \mathbb{R}^3$ and we have $V \cap W = \{0_{\mathbb{R}^3}\}$, thus $\mathbb{R}^3 = V \oplus W$

Example 27. (24) Let V and W be two subspaces of \mathbb{R}^3 such that

$$V = \left\{ \begin{pmatrix} 0 \\ c \\ d \end{pmatrix} : c, d \in \mathbb{R} \right\}, \quad W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Solution : We have,

$$V + W = \left\{ \begin{pmatrix} a \\ b + c \\ d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

We know from the previous example (24) that $\mathbb{R}^3 \neq V \oplus W$. Let's make sure the property $V \cap W \neq \{0_{\mathbb{R}^3}\}$ is correct.

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, then

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \cap W &\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W, \\ &\Leftrightarrow \begin{cases} x = 0 \\ y = \alpha \\ z = 0 \end{cases} \quad \alpha \in \mathbb{R} \end{aligned}$$

For example $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \in \mathbb{V} \cap \mathbb{W}$, thus $\mathbb{V} \cap \mathbb{W} \neq \{0_{\mathbb{R}^3}\}$.

Definition 1.1.12. *The vector space \mathbb{V} is the direct sum of its subspaces \mathbb{U} and \mathbb{W} if $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$.*

In the finite-dimensional case :

$$\mathbb{V} = \mathbb{U} \oplus \mathbb{W} \Leftrightarrow \dim(\mathbb{V}) = \dim(\mathbb{U}) + \dim(\mathbb{W})$$

Linear maps

2.1 Linear maps (Linear transformations)

Definition 2.1.1. Let \mathbb{U} and \mathbb{V} be two vectors spaces over a field \mathbb{F} . A function $T : \mathbb{U} \rightarrow \mathbb{V}$ is called a **linear map** or **linear transformations** if it satisfies

1. $\forall u, v \in \mathbb{U}, T(u + v) = T(u) + T(v)$ T is additive or T preserves addition
2. $\forall u \in \mathbb{U}, \forall \alpha \in \mathbb{F} T(\alpha u) = \alpha T(u)$ T is preserves scalar multiplication

- An **operator** on a vector space \mathbb{U} is a linear map from \mathbb{U} to itself
- The set of all linear transformations from \mathbb{U} to \mathbb{V} is denoted $\mathcal{L}(\mathbb{U}, \mathbb{V})$. Also, $\mathcal{L}(\mathbb{U}, \mathbb{U}) = \mathcal{L}(\mathbb{U})$ and $T \in \mathcal{L}(\mathbb{U})$ is called a linear transformation on \mathbb{U} .

Example 28.

- The zero function $\forall u \in \mathbb{U}, 0(u) = 0_{\mathbb{V}}$ is a linear map it called **trivial transformation** or **trivial linear map**
- The identity function $\forall u \in \mathbb{U}, I_d(u) = u$ is a linear map

Example 29. Let

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (2x + y, -y)$$

Let's verify That the map T is indeed a linear map.

1. $\forall u = (x, y), v = (x', y') \in \mathbb{R}^2,$

$$\begin{aligned} T(u + v) &= T((x + x', y + y')) = (2(x + x') + (y + y'), -(y + y')) \\ &= ((2x + y) + (2x' + y'), -y - y') \\ &= (2x + y, -y) + (2x' + y', -y') \\ &= T(u) + T(v) \end{aligned}$$

2. $\forall u = (x, y) \in \mathbb{R}^2, \forall \alpha \in \mathbb{R},$

$$\begin{aligned} T(\alpha u) &= (\alpha(x, y)) = (\alpha x, \alpha y) \\ &= (2\alpha x + \alpha y, -\alpha y) = \alpha(2x + y, -y) \\ &= \alpha T(u) \end{aligned}$$

Example 30. *Let*

$$\begin{aligned} T : \mathbb{R}_3[x] &\longrightarrow \mathbb{R}_3[x] \\ P(x) &\longmapsto T(P(x)) = P'(x) \end{aligned}$$

The map T is linear since

1.

$$\begin{aligned} \forall p(x), q(x) \in \mathbb{R}_3[x], \quad T(p(x) + q(x)) &= (p(x) + q(x))' = p'(x) + q'(x) \\ &= T(p(x)) + T(q(x)) \end{aligned}$$

2.

$$\forall p(x) \in \mathbb{R}_3[x], \forall \alpha \in \mathbb{R}, \quad T(\alpha p(x)) = (\alpha p(x))' = \alpha p'(x) = \alpha T(p(x)).$$

Example 31. *Let*

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, 1) \end{aligned}$$

This is not a linear map.

Theorem 2.1.1. *If $T : \mathbb{U} \longrightarrow \mathbb{V}$ is a linear map then :*

1. $T(0_{\mathbb{U}}) = 0_{\mathbb{V}}$
2. $T(-u) = -T(u), \quad \forall u \in \mathbb{U}$

Proof :

We know by definition that : $\forall u \in \mathbb{U}, \forall \alpha \in \mathbb{F}, T(\alpha u) = \alpha T(u)$. In particular,
— if $\alpha = 0_{\mathbb{F}}$.

$$T(0_{\mathbb{U}}) = T(0_{\mathbb{F}}u) = 0_{\mathbb{F}}T(u) = 0_{\mathbb{V}}$$

— if $\alpha = -1$.

$$T(-u) = -T(u)$$

Theorem 2.1.2. *$T : \mathbb{U} \longrightarrow \mathbb{V}$ is a linear map if and only if*

$$\forall u, v \in \mathbb{U}, \forall \alpha \in \mathbb{F}, \quad T(\alpha u + v) = \alpha T(u) + T(v)$$

Proof : Easy to proof it

Theorem 2.1.3. *Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$. If S is linearly dependent then $T(S)$ is linearly dependent*

Proof :

Suppose there exists $\alpha_1, \alpha_2, \dots, \alpha_p \in \mathbb{F}$ for which $\sum_1^p \alpha_i v_i$ is a linear dependence in S . Calculate,

$$T\left(\sum_1^p \alpha_i v_i\right) = \sum_1^p \alpha_i T(v_i).$$

2.1.1 kernel and Image of a linear map

Definition 2.1.2. Let \mathbb{U}, \mathbb{V} be vector spaces over a field \mathbb{F} and $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$ then

$$\ker(T) = \{u \in \mathbb{U} \mid T(u) = 0_{\mathbb{V}}\} \quad \text{and} \quad \text{Im}(T) = \{T(u) \in \mathbb{V} \mid u \in \mathbb{U}\}$$

Example 32. Let

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto T((x, y)) = (x + 2y, -y) \end{aligned}$$

$$\begin{aligned} \ker(T) &= \{u \in \mathbb{R}^2 \mid T(u) = 0_{\mathbb{R}^2}\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid (x + 2y, -y) = (0, 0)\} \\ &= \{(0, 0)\} \end{aligned}$$

$$\begin{aligned} \text{Im}(T) &= \{T(u) \in \mathbb{R}^2 \mid u \in \mathbb{R}^2\} \\ &= \{(x + 2y, -y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2\} \end{aligned}$$

We have

$$\begin{aligned} (z, t) \in \text{Im}(T) &\Leftrightarrow \exists x, y \in \mathbb{R} \text{ s.t.; } (z, t) = (x + 2y, -y) \\ &\Leftrightarrow \begin{cases} z = x + 2y \\ t = -y \end{cases} \quad x, y \in \mathbb{R} \\ &\Leftrightarrow \begin{cases} x = z - 2y \\ y = -t \end{cases} \end{aligned}$$

Therefore $\text{Im}(T) = \mathbb{R}^2$

Example 33. Let

$$\begin{aligned} T : \mathbb{R}_3[x] &\longrightarrow \mathbb{R}_3[x] \\ P(x) &\longmapsto T(P(x)) = P'(x) \end{aligned}$$

$$\begin{aligned} \text{Ker}(T) &= \{p(x) \in \mathbb{R}_3[x] \mid T(p(x)) = 0_{\mathbb{R}_3[x]}\} \\ &= \{p(x) \in \mathbb{R}_3[x] \mid p'(x) = 0_{\mathbb{R}_3[x]} = 0\} \\ &= \{p(x) \in \mathbb{R}_3[x] \mid p(x) = \text{constant polynomials}\} \end{aligned}$$

$$\begin{aligned}
\text{Im}(T) &= \{T(p(x)) \in \mathbb{R}_3[x] \mid p(x) \in \mathbb{R}_3[x]\} \\
&= \{p'(x) \in \mathbb{R}_3[x] \mid p(x) \in \mathbb{R}_3[x]\} \\
&= \mathbb{R}_3[x]
\end{aligned}$$

◇ **Some geometric examples**

Example 34.

1. Let

$$\begin{aligned}
P_{(xy)} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto P_{(xy)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}
\end{aligned}$$

Verify that $P_{(xy)}$ is linear map, $P_{(xy)}$ is called a **projection on the plane** (xy) and we have

$$\text{Ker}(P_{(xy)}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, z \in \mathbb{R} \right\}, \quad \text{Im}(P_{(xy)}) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, x, y \in \mathbb{R} \right\}$$

2. Let

$$\begin{aligned}
P_{(xz)} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto P_{(xz)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}
\end{aligned}$$

Verify that $P_{(xz)}$ is linear map, $P_{(xz)}$ is called a **projection on the plane** (xz) and we have

$$\text{Ker}(P_{(xz)}) = \left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, z \in \mathbb{R} \right\}, \quad \text{Im}(P_{(xz)}) = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}, x, y \in \mathbb{R} \right\}$$

3. Let

$$\begin{aligned}
P_{(yz)} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto P_{(yz)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}
\end{aligned}$$

Verify that $P_{(yz)}$ is linear map, $P_{(yz)}$ is called a **projection on the plane** (yz) and we have

$$\text{Ker}(P_{(yz)}) = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, z \in \mathbb{R} \right\}, \quad \text{Im}(P_{(yz)}) = \left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}, x, y \in \mathbb{R} \right\}$$

4. Let

$$\begin{aligned}
\text{Ref}_{(yz)} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto \text{Ref}_{(yz)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}
\end{aligned}$$

Verify that $Ref_{(yz)}$ is linear map, $Ref_{(xy)}$ is called a **reflection with respect to the** (yz) plane and we have

$$Ker(Ref_{(xy)}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad Im(Ref_{(xy)}) = \mathbb{R}^3$$

Similarly we define the reflection with respect to the (xz) and (xy) planes

5. Let

$$Rot_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto Rot_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix}, \quad 0 \leq \theta \leq 2\pi$$

Verify that Rot_θ is linear map, Rot_θ is called a **rotation** it rotates by θ counter-clockwise and we have

$$Ker(Rot_\theta) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad Im(Rot_\theta) = \mathbb{R}^2$$

Remark 2.1.1.

- a) If $\theta = 0$, we have $Rot_0 = Id_{\mathbb{R}^2}$ identity map on plane
- b) If $\theta = \pi$, we have $Rot_\pi = Ref_{(0)}$ reflection with respect to the origin

Properties 2.1.1. Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$,

1. $Ker(T)$ is a subspace of \mathbb{U}
2. $Im(T)$ is a subspace of \mathbb{V}
3. T is **one-to-one** (Injective) if and only if : $Ker(T) = \{0_{\mathbb{U}}\}$
4. T is **onto** (surjective) if and only if : $Image(T) = \mathbb{V}$

Proof :

1. $Ker(T)$ is a subspace of \mathbb{U} ? . We have

- $T(0_{\mathbb{U}}) = 0_{\mathbb{V}}$ thus $0_{\mathbb{U}} \in Ker(T) \Rightarrow Ker(T) \neq \emptyset$
- Let $u, v \in Ker(T)$ and $\alpha \in \mathbb{F}$, we have

$$T(\alpha u + v) = \alpha \underbrace{T(u)}_{=0_{\mathbb{V}}} + \underbrace{T(v)}_{=0_{\mathbb{V}}} = 0_{\mathbb{V}} \Rightarrow (\alpha u + v) \in Ker(T)$$

Thus $Ker(T)$ is a subspace of \mathbb{U} .

2. $Im(T)$ is a subspace of \mathbb{V} ? We have

- $T(0_{\mathbb{U}}) = 0_{\mathbb{V}}$ thus $0_{\mathbb{V}} \in Im(T) \Rightarrow Im(T) \neq \emptyset$
- Let $w, t \in Im(T)$ and $\alpha \in \mathbb{F}$, we have

$$w, t \in Im(T) \Rightarrow \exists u, v \in \mathbb{U}, w = T(u), \text{ and } t = T(v)$$

then

$$\alpha w + t = \alpha T(u) + T(v) = T(\underbrace{\alpha u + v}_{\in \mathbb{U}}), \text{ Because } \mathbb{U} \text{ is V-S and } T \text{ is L-Map}$$

So, $\alpha w + t \in \text{Im}(T)$. Thus $\text{Im}(T)$ is a subspace of \mathbb{V} .

3. T is **one-to-one** (Injective) if and only if : $\text{Ker}(T) = \{0_{\mathbb{U}}\}$?

— First we will provethat : $\text{Ker}(T) = \{0_{\mathbb{U}}\} \Rightarrow T$ is one-to-one.

Let $\text{Ker}(T) = \{0_{\mathbb{U}}\}$ and $u, v \in \mathbb{U}$ s.t $T(u) = T(v)$, then

$$\begin{aligned} T(u) = T(v) &\Rightarrow T(u) - T(v) = 0_{\mathbb{V}} \\ &\Rightarrow T(u - v) = 0_{\mathbb{V}} \\ &\Rightarrow (u - v) \in \text{Ker}(T) \\ &\Rightarrow u - v = 0_{\mathbb{U}} \\ &\Rightarrow u = v \Rightarrow T \text{ is one-to-one.} \end{aligned}$$

— Now we will provethat : T is one-to-one $\Rightarrow \text{Ker}(T) = \{0_{\mathbb{U}}\}$.

It is known that T is one-to-one if and only if

$$\forall u, v \in \mathbb{U}, T(u) = T(v) \Rightarrow u = v$$

Let T be one-to-one map, we have.

$$u \in \text{Ker}(T) \Rightarrow T(u) = 0_{\mathbb{V}} = T(0_{\mathbb{U}}) \Rightarrow u = 0_{\mathbb{U}}$$

Thus $\text{Ker}(T) = \{0_{\mathbb{U}}\}$.

4. T is **onto** (surjective) if and only if : $\text{Im}(T) = \mathbb{V}$?

— First we will provethat : $\text{Im}(T) = \mathbb{V} \Rightarrow T$ is onto map.

Let $\text{Im}(T) = \mathbb{V}$ that mains $\forall w \in \mathbb{V} = \text{Im}(T), \exists u \in \mathbb{U}, \text{ st } T(u) = w$. Thus T is onto map.

— Now we will provethat : T is onto $\Rightarrow \text{Im}(T) = \mathbb{V}$.

It is known that : $\text{Im}(T) \subset \mathbb{V} \dots (1)$ and T is onto map if and only if

$$\forall w \in \mathbb{V}, \exists u \in \mathbb{U}, w = T(u).$$

Let T be onto map, we have.

$$\forall w \in \mathbb{V} \Rightarrow \exists u \in \mathbb{U}, w = T(u) \Rightarrow w \in \text{Im}(T) \dots (2)$$

From (1) and (2), we get $\text{Im}(T) = \mathbb{V}$.

Theorem 2.1.4. Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$:

1. If \mathbb{W} is subspace of \mathbb{U} , then $T(\mathbb{W})$ is subspace of \mathbb{V} .
2. If \mathbb{H} is subspace of \mathbb{V} , then $T^{-1}(\mathbb{H})$ is subspace of \mathbb{U}

Proof : Easy

Theorem 2.1.5. Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$:

If $\{u_1, u_2, \dots, u_p\}$ spans \mathbb{U} then $\{T(u_1), T(u_2), \dots, T(u_p)\}$ spans $Im(T) \subset \mathbb{V}$.

Proof : Let $\mathbb{U} = span \{u_1, u_2, \dots, u_p\}$, and $w \in \mathbb{V}$

$$\begin{aligned} w \in Im(T) &\Leftrightarrow \exists u_i \in \mathbb{U}, \exists \alpha_i \in \mathbb{F}, (i = 1, 2, \dots, p), w = T \left(\sum_1^p \alpha_i u_i \right) \\ &\Leftrightarrow w = \sum_1^p \alpha_i T(u_i), u_i \in \mathbb{U}, \alpha_i \in \mathbb{F} (i = 1, 2, \dots, p) \end{aligned}$$

Thus : $Im(T) = span \{T(u_1), T(u_2), \dots, T(u_p)\}$.

Example 35. Let

$$\begin{aligned} T : \mathbb{R}^4 &\longrightarrow \mathbb{R}_2[x] \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &\longmapsto (a+b) + (c+2d)x + (2c+4d)x^2 \end{aligned}$$

Verify that T is linear map

$$Ker(T) = \left\{ u = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \mid T(u) = 0_{\mathbb{R}_2[x]} \right\}$$

$$\begin{aligned} T(u) = 0_{\mathbb{R}_2[x]} &\Leftrightarrow (a+b) + (c+2d)x + (2c+4d)x^2 = 0 \\ &\Leftrightarrow \begin{cases} a+b = 0 \\ c+2d = 0 \\ 2c+4d = 0 \end{cases} \Leftrightarrow a = 1, b = -1, c = 1, d = -1/2 \end{aligned}$$

Thus

$$\begin{aligned}
 \text{Ker}(T) &= \left\{ \begin{pmatrix} a \\ -a \\ c \\ -(1/2)c \end{pmatrix} \mid a, c \in \mathbb{R} \right\} \\
 &= \left\{ a \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ -(1/2) \end{pmatrix} \mid a, c \in \mathbb{R} \right\} \\
 &= \text{span} \left\{ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -(1/2) \end{pmatrix}}_{u_2} \right\}
 \end{aligned}$$

u_1 and u_2 are linear independent and thus basis therefore $\dim(\text{Ker}(T)) = 2$.

Let's find $\text{Im}(T)$: It's known that the basis of \mathbb{R}^4 is :

$$\mathcal{B} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{e_4} \right\}$$

T of these ill give a spanning set for the subspace $\text{Im}(T)$.

$$\begin{aligned}
 T(\mathcal{B}) &= \{T(e_1), T(e_2), T(e_3), T(e_4)\} \\
 &= \{1, 1, x + 2x^2, 2x + 4x^2\} \\
 &= \{1, x + 2x^2\} \quad \text{show that } x + 2x^2, 2x + 4x^2 \text{ are L-dependent}
 \end{aligned}$$

Thus

$$\text{Im}(T) = \text{span} \{1, x + 2x^2\}$$

and $\{1, x + 2x^2\}$ is basis for $\text{Im}(T)$. which get $\dim(\text{Im}(T)) = 2$

2.1.2 Rank of linear map

Definition 2.1.3. Let T be a linear map. The rank of T is a positive real : $r(T) = \dim(\text{Im}(T))$

Example 36. Let (previous example)

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}_2[x]$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longmapsto (a+b) + (c+2d)x + (2c+4d)x^2$$

The rank of T is $r(T) = \dim(\text{Im}(T)) = 2$

Proposition 2.1.1. Let $\mathcal{L}(\mathbb{U}, \mathbb{V})$, then :

$$\dim(\mathbb{U}) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$$

Proof : Denote $n = \dim(\mathbb{U})$, $k = \dim(\text{Ker}(T))$, we'll prove that $\dim(\text{Im}(T)) = n - k$.

Let $\{u_1, u_2, \dots, u_k\}$ be a basis for $\text{Ker}(T)$ and $n = \dim(\mathbb{U})$, $k = \dim(\text{Ker}(T))$ for $\text{Ker}(T)$. Complete it to form a basis for \mathbb{U} ,

$$\{u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_n\}$$

We know that

$$T(u_1) = T(u_2) = \dots = T(u_k) = 0_{\mathbb{V}}, \quad \text{since } u_1, u_2, \dots, u_k \in \text{Ker}(T).$$

$$\text{Also, } \left\{ \underbrace{T(u_1)}_{=0_{\mathbb{V}}}, \underbrace{T(u_2)}_{=0_{\mathbb{V}}}, \dots, \underbrace{T(u_k)}_{=0_{\mathbb{V}}}, T(u_{k+1}) \dots, T(u_n) \right\} \text{ spans } \text{Im}(T), \text{ which leads}$$

$$\{T(u_{k+1}), T(u_{k+2}) \dots, T(u_n)\}, \quad \text{spans } \text{Im}(T)$$

In order to complete the proof, we need to show that it is linear independent.

Let $\alpha_i \in \mathbb{F}$, ($i = (k+1), (k+2), \dots, n$)

$$\begin{aligned} \alpha_{k+1}T(u_{k+1}) + \alpha_{k+2}T(u_{k+2}) + \dots + \alpha_n T(u_n) = 0_{\mathbb{V}} &\Rightarrow T(\alpha_{k+1}u_{k+1}) + T(\alpha_{k+2}u_{k+2}) + \dots + T(\alpha_n u_n) = 0_{\mathbb{V}} \\ &\Rightarrow T(\alpha_{k+1}u_{k+1} + \alpha_{k+2}u_{k+2} + \dots + \alpha_n u_n) = 0_{\mathbb{V}} \\ &\Rightarrow (\alpha_{k+1}u_{k+1} + \alpha_{k+2}u_{k+2} + \dots + \alpha_n u_n) \in \text{Ker}(T) \end{aligned}$$

That means a vector $\alpha_{k+1}u_{k+1} + \alpha_{k+2}u_{k+2} + \dots + \alpha_n u_n$ is a combination of basis for $\text{Ker}(T)$

$$\alpha_{k+1}u_{k+1} + \alpha_{k+2}u_{k+2} + \dots + \alpha_n u_n = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k, \quad \text{for some } \alpha_1, \dots, \alpha_k$$

Therefore

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k - \alpha_{k+1}u_{k+1} - \alpha_{k+2}u_{k+2} - \dots - \alpha_n u_n = 0_{\mathbb{V}}$$

All α 's and in particular $\alpha_{k+1}u_{k+1}, \alpha_{k+2}u_{k+2}, \dots, \alpha_n u_n$ are $0_{\mathbb{F}}$ Then,

$$\left\{ \underbrace{T(u_1)}_{=0_{\mathbb{V}}}, \underbrace{T(u_2)}_{=0_{\mathbb{V}}}, \dots, \underbrace{T(u_k)}_{=0_{\mathbb{V}}}, T(u_{k+1}) \dots, T(u_n) \right\}$$

is linear independent and thus a basis for $Im(T)$ with $n - k$ vectors so, $dim(Im(T)) = n - k$.

Corollary 2.1.1. *Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$ is one-to-one and onto (bijection) then $dim(\mathbb{U}) = dim(\mathbb{V})$*

Proof : We know that :

$$T \text{ is one-to-one} \Leftrightarrow Ker(T) = \{0_{\mathbb{V}}\} \Leftrightarrow dim(Ker(T)) = 0$$

$$T \text{ is onto} \Leftrightarrow Im(T) = \mathbb{V} \Leftrightarrow dim(Im(T)) = dim(\mathbb{V})$$

and

$$\begin{aligned} dim(\mathbb{U}) &= dim(Ker(T)) + dim(Im(T)) \\ &= 0 + dim(\mathbb{V}) = dim(\mathbb{V}) \end{aligned}$$

Corollary 2.1.2. *Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$. If $dim(\mathbb{U}) = dim(\mathbb{V})$ then :*

$$T \text{ is one-to-one} \Leftrightarrow T \text{ is onto}$$

Proof :

— Let : $dim(\mathbb{U}) = dim(\mathbb{V})$ and T be one-to-one map ($dim(Ker(T)) = 0$).

We have :

$$dim(\mathbb{U}) = \underbrace{dim(Ker(T))}_{=0} + dim(Im(T))$$

but : $dim(\mathbb{U}) = dim(\mathbb{V})$, thus $Im(T) = \mathbb{V}$ which leads T is onto map.

— Let : $dim(\mathbb{U}) = dim(\mathbb{V})$ and T be onto map ($Im(T) = \mathbb{V}$).

We have :

$$dim(\mathbb{U}) = dim(Ker(T)) + \underbrace{dim(Im(T))}_{=dim(\mathbb{V})}$$

but : $dim(\mathbb{U}) = dim(\mathbb{V})$, thus $dim(Ker(T)) = 0$ or $Ker(T) = \{0_{\mathbb{U}}\}$ which leads T is one-to-one map.

2.1.3 Operations on linear maps

Theorem 2.1.6. *If $T, S \in \mathcal{L}(\mathbb{U}, \mathbb{V})$ then $T + S, \alpha T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$, $\alpha \in \mathbb{F}$.*

Proof Easy to proof it

Example 37. *Let*

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \quad \text{is linear map} \\ (x, y) &\longmapsto T((x, y)) = (x + 2y, -y) \end{aligned}$$

$$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ is linear map}$$

$$(x, y) \longmapsto S((x, y)) = (3x - 2y, x + 3y)$$

we have

$$(T + S) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ is linear map}$$

$$(x, y) \longmapsto (T + S)((x, y)) = (4x, x + 2y)$$

Also

$$\alpha T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ is linear map}$$

$$(x, y) \longmapsto (\alpha T)((x, y)) = (\alpha x + 2\alpha y, -\alpha y)$$

2.1.4 Composition of linear maps

Definition 2.1.4. Let $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$ and $S \in \mathcal{L}(\mathbb{V}, \mathbb{W})$, we define the map $S \circ T$ as

$$(S \circ T) : \mathbb{U} \longrightarrow \mathbb{W}$$

$$u \longmapsto (S \circ T)(u) = S(T(u))$$

$S \circ T$ is the **composition** of S and T ,

Claim 2.1. $S \circ T$ is linear map

Proof Let $u, v \in \mathbb{U}$ and $\alpha \in \mathbb{F}$, we have

$$(S \circ T)(\alpha u + v) = S(T(\alpha u + v)) = S(\alpha T(u) + T(v)), \text{ because } T \text{ is linear}$$

$$= \alpha S(T(u)) + S(T(v)), \text{ because } S \text{ is linear}$$

$$= \alpha(S \circ T)(u) + (S \circ T)(v), \text{ by definition.}$$

Thus $S \circ T$ is linear map.

Example 38. Let

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longmapsto T(x, y) = (x + 2y, -y)$$

$$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longmapsto S(x, y) = (3y, x - y)$$

we have

$$(TS)(x, y) = T(S(x, y)) = T(3y, x - y)$$

$$= (3y + 2(x - y), -(x - y))$$

$$= (2x + y, y - x)$$

Thus

$$(TS) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longmapsto (TS)(x, y) = (2x + y, y - x)$$

Example 39. *Let*

$$S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \longmapsto \begin{pmatrix} x + y \\ 0 \\ z + t \\ 0 \end{pmatrix}$$

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}_3[t]$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longmapsto b + dt^2 + (b + d)t^3$$

we have

$$\begin{aligned} (T \circ S) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= T \left(S \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right) \\ &= T \begin{pmatrix} a + b \\ 0 \\ c + d \\ 0 \end{pmatrix} \\ &= 0 + 0t^2 + 0t^3 = 0 \end{aligned}$$

Thus $T \circ S$ is the zero map.

Example 40. *Let*

$$T = Proj : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \text{Projection}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$S = Ref : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \text{Reflection}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

We have

$$(T \circ S) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \left(S \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = T \begin{pmatrix} -x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix}$$

Also

$$(S \circ T) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = S \left(T \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = S \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix}$$

In this example we get $S \circ T = T \circ S$, but **in general** $S \circ T \neq T \circ S$

2.1.5 Inverse of a map

Definition 2.1.5. Let $T \in \mathcal{L}(\mathbb{U})$ be one-to-one and onto (bijection) as a function T has an inverse, denoted T^{-1} satisfying

$$T \circ T^{-1} = T^{-1} \circ T = Id_{\mathbb{U}}$$

Claim 2.2. T^{-1} is a linear map

Proof : Let $T \in \mathcal{L}(\mathbb{U})$ and $\alpha \in \mathbb{F}$, so let $u, v, w, t \in \mathbb{U}$ such that $w = T(u)$, and $t = T(v)$,

We have :

$$w = T(u) \Leftrightarrow T^{-1}(w) = u, \quad t = T(v) \Leftrightarrow T^{-1}(t) = v$$

Then,

$$\begin{aligned} T^{-1}(\alpha w + t) &= T^{-1}(\alpha T(u) + T(v)), \quad \text{such that } w = T(u), \quad t = T(v) \\ &= T^{-1}(T(\alpha u + v)), \quad \text{because } T \text{ is linear} \\ &= \underbrace{(T^{-1}T)}_{Id_{\mathbb{U}}}(\alpha u + v), \quad \text{by definition} \\ &= \alpha u + v = \alpha T^{-1}(w) + T^{-1}(t) \end{aligned}$$

Therefore T^{-1} is linear

Example 41.

$Rot_{\theta} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is linear and one-to-one and onto

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto Rot_{\theta} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix}, \quad 0 \leq \theta \leq 2\pi$$

The inverse of a rotation Rot_{θ} is a rotation $Rot_{(-\theta)}$, i.e. $Rot_{\theta}^{-1} = Rot_{(-\theta)}$

$$Rot_{\theta}^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto Rot_{\theta}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \theta + y \sin \theta \\ -x \sin \theta + y \cos \theta \end{pmatrix}, \quad 0 \leq \theta \leq 2\pi$$

It is easy to see that $Rot_{\theta}^{-1}Rot_{\theta} = Id_{\mathbb{R}^2}$ (where $Rot_{\theta}^{-1}Rot_{\theta} = Rot_{\theta}^{-1} \circ Rot_{\theta}$), in fact

$$\begin{aligned}
 (Rot_{\theta}^{-1}Rot_{\theta}) \begin{pmatrix} x \\ y \end{pmatrix} &= Rot_{\theta}^{-1} \left(Rot_{\theta} \begin{pmatrix} x \\ y \end{pmatrix} \right) \\
 &= Rot_{\theta}^{-1} \left(\begin{pmatrix} \underbrace{x \cos \theta - y \sin \theta}_{x'} \\ \underbrace{x \sin \theta + y \cos \theta}_{y'} \end{pmatrix} \right) \\
 &= \begin{pmatrix} x' \cos \theta + y' \sin \theta \\ -x' \sin \theta + y' \cos \theta \end{pmatrix} \\
 &= \begin{pmatrix} (x \cos \theta - y \sin \theta) \cos \theta + (x \sin \theta + y \cos \theta) \sin \theta \\ -(x \cos \theta - y \sin \theta) \sin \theta + (x \sin \theta + y \cos \theta) \cos \theta \end{pmatrix} \\
 &= \begin{pmatrix} x(\cos^2 \theta + \sin^2 \theta) \\ y(\cos^2 \theta + \sin^2 \theta) \end{pmatrix} \\
 &= \begin{pmatrix} x \\ y \end{pmatrix}
 \end{aligned}$$

Thus $Rot_{\theta}^{-1}Rot_{\theta} = Id_{\mathbb{R}^2}$ (Similarly $Rot_{\theta}Rot_{\theta}^{-1} = Id_{\mathbb{R}^2}$).

Remark :

$$Rot_{\theta}Rot_{\theta'} = Rot_{(\theta+\theta')}.$$

Example 42.

$$\begin{aligned}
 T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
 \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -y \end{pmatrix}
 \end{aligned}$$

T is linear and one-to-one and onto. Find T^{-1}

Solution : We have

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -y \end{pmatrix}$$

Let

$$\begin{pmatrix} w \\ t \end{pmatrix} = \begin{pmatrix} x + 2y \\ -y \end{pmatrix}$$

we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} w + 2t \\ -t \end{pmatrix}$$

Thus

$$\begin{aligned}
 T^{-1} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
 \begin{pmatrix} w \\ t \end{pmatrix} &\longmapsto T^{-1} \begin{pmatrix} w \\ t \end{pmatrix} = \begin{pmatrix} w + 2t \\ -t \end{pmatrix}
 \end{aligned}$$

Theorem 2.1.7. *Let $T \in \mathcal{L}(\mathbb{U})$ then*

$$(T \text{ is invertible}) \Leftrightarrow (T \text{ is onto}) \Leftrightarrow (\text{is one-to-one})$$

Proof : we had a theorem that if $\dim(\mathbb{U}) = \dim(\mathbb{V})$ then

$$(T \text{ is onto}) \Leftrightarrow (\text{is one-to-one})$$

Theorem 2.1.8. *If T, S are invertible then $T \circ S$ is invertible, and*

$$(T \circ S)^{-1} = S^{-1} \circ T^{-1}$$

Proof The proof comes from the properties of functions composition. We have

$$(T \circ S) \circ (S^{-1} \circ T^{-1}) = T \circ (S \circ S^{-1}) \circ T^{-1} = T \circ (Id_{\mathbb{U}}) \circ T^{-1} = T \circ T^{-1} = Id_{\mathbb{U}}, \quad (\circ \text{ Ass})$$

and

$$(S^{-1} \circ T^{-1}) \circ (T \circ S) = S^{-1} \circ (T \circ T^{-1}) \circ S = S^{-1} \circ (Id_{\mathbb{U}}) \circ S = S^{-1} \circ S = Id_{\mathbb{U}}$$

Thus $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$

Exercise 15. *Let T be a map such that*

$$\begin{aligned} T : \mathbb{R}_2[x] &\longrightarrow \mathbb{R}[x] \\ (a + bx + cx^2) &\longmapsto T((a + bx + cx^2)) = a(x + 1) + cx + b \end{aligned}$$

1. Prove that a map T is linear
2. Find $\text{Ker}(T)$ and $\text{Im}(T)$
3. Find $\text{Rank}(T)$

Solution 15.

1. For polynomials $p = a + bx + cx^2$, $q = a' + b'x + c'x^2$ and scalar λ :

$$T(p + q) = (a + a')(x + 1) + (c + c')x + (b + b') = T(p) + T(q),$$

$$T(\lambda p) = \lambda a(x + 1) + \lambda cx + \lambda b = \lambda T(p).$$

2. Write

$$T(p) = ax + a + cx + b = (a + c)x + (a + b).$$

So

$$T(p) = \alpha x + \beta \quad \text{with} \quad \alpha = a + c, \quad \beta = a + b.$$

Kernel :

$$\text{Ker}T = \{p \in \mathbb{R}_2[x], p = 0\}$$

thus

$$p = 0 \Rightarrow a + c = 0 \quad \text{and} \quad a + b = 0.$$

Hence

$$b = -a, \quad c = -a.$$

So

$$p = a - ax - ax^2 = a(1 - x - x^2).$$

Thus

$$\ker(T) = \text{span}\{1 - x - x^2\},$$

and

$$\dim(\ker(T)) = 1.$$

3. **Image** : any linear polynomial $rx + s$ can be obtained. We need $a + c = r$ and $a + b = s$. Choose $a = 0$, then $c = r$, $b = s$ gives

$$p = sx + rx^2.$$

So image is all of $\mathbb{R}_1[x]$ (degree ≤ 1). Thus

$$\text{Im } T = \mathbb{R}_1[x]$$

and

$$\dim(\text{Im } T) = 2.$$

we can use the rank theorem

$$\underbrace{\dim(\mathbb{R}_2[x])}_{=3} = \underbrace{\dim(\ker(T))}_{=1} + \dim(\text{Im } T) \Rightarrow \dim(\text{Im } T) = 3 - 1 = 2$$

4.

$$\text{rank } T = \dim(\text{Im } T) = 2.$$

2.1.6 Isomorphism

Definition 2.1.6. Let \mathbb{U} and \mathbb{V} be two vectors spaces over \mathbb{F} .

\mathbb{U} and \mathbb{V} are called **isomorphic** if there exists a linear map $T : \mathbb{U} \rightarrow \mathbb{V}$ which is one-to-one and onto. T is called an isomorphism

Notation : in this case we write $\mathbb{U} \cong \mathbb{V}$

Example 43.

$$T : \mathbb{R}^4 \rightarrow \mathbb{R}_3[x]$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a + bx + cx^2 + dx^3$$

T is linear, one-to-one and onto, thus $\mathbb{R}^4 \cong \mathbb{R}_3[x]$.

Claim 2.3. Using the linear map :

$$T : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}_n[x]$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_{(n+1)} \end{pmatrix} \longmapsto T \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_{(n+1)} \end{pmatrix} = a_1 + a_2x + a_3x^2 + \cdots + a_{(n+1)}x^n$$

we can prove that

$$\mathbb{R}_n[x] \cong \mathbb{R}^{n+1}$$

Remark 2.1.2. Isomorphism of vector spaces is an equivalence relation.

1. Reflexive : $\mathbb{U} \cong \mathbb{U}$
2. Symmetric : $\mathbb{U} \cong \mathbb{V} \Rightarrow \mathbb{V} \cong \mathbb{U}$
3. Transitive : $\mathbb{U} \cong \mathbb{V}, \mathbb{V} \cong \mathbb{W} \Rightarrow \mathbb{U} \cong \mathbb{W}$

Theorem 2.1.9.

$$\mathbb{U} \cong \mathbb{V} \Leftrightarrow \dim(\mathbb{U}) = \dim(\mathbb{V})$$

Proof :

1. (\Leftarrow) Suppose that : $\dim(\mathbb{U}) = \dim(\mathbb{V})$ and let $\{u_1, u_2, \dots, u_n\}$ be a basis for \mathbb{U} and $\{v_1, v_2, \dots, v_n\}$ be a basis for \mathbb{V} . Then

$$T(\alpha_1u_1 + \alpha_2u_2 + \cdots + \alpha_nu_n) = \alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_nv_n$$

is linear, one-to-one and onto by a previous theorem, and hence implements an isomorphism $T : \mathbb{U} \longrightarrow \mathbb{V}$

2. (\Rightarrow) Suppose that : $\mathbb{U} \cong \mathbb{V}$ and let $\{u_1, u_2, \dots, u_n\}$ be a basis for \mathbb{U} , denote

$$A = \{T(u_1), T(u_2), \dots, T(u_n)\} \subseteq \mathbb{V}$$

where T is an isomorphism $T : \mathbb{U} \longrightarrow \mathbb{V}$.

By a previous theorem A spans $Im(T) = \mathbb{V}$ (since T is onto). A is linear independent. So, if $\alpha_1T(u_1) + \alpha_2T(u_2) + \cdots + \alpha_nT(u_n) = 0_{\mathbb{V}}$ then $T(\alpha_1u_1 + \alpha_2u_2 + \cdots + \alpha_nu_n) = 0_{\mathbb{V}}$ so $\alpha_1u_1 + \alpha_2u_2 + \cdots + \alpha_nu_n = 0_{\mathbb{U}}$ (since T is one-to-one $Ker(T) = \{0_{\mathbb{U}}\}$). and thus $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0_{\mathbb{F}}$

Corollary 2.1.3. Every vector space of dimension n over \mathbb{F} is isomorphic to \mathbb{F}^n

$$\mathbb{U} \text{ is vector space over } \mathbb{F} \Rightarrow \mathbb{U} \cong \mathbb{F}^n$$

Proposition 2.1.2. Let \mathbb{U} be a vector space and $\mathcal{B} = \{e_i\}_{1 \leq i \leq n}$ a basis of \mathbb{U} then,

$$\forall u \in \mathbb{U}, \exists (x_i)_{1 \leq i \leq n} \in \mathbb{F} : u = \sum_{i=1}^{i=n} x_i e_i$$

The scalars x_i are called the components of u , and we write :

$$u = [u]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

2.1.7 Hom(\mathbb{U} , \mathbb{V})

Definition 2.1.7. Let \mathbb{U} and \mathbb{V} be two vectors spaces over \mathbb{F} . The collection of all linear maps from \mathbb{U} and \mathbb{V} is called $Hom(\mathbb{U}, \mathbb{V})$

Theorem 2.1.10. $Hom(\mathbb{U}, \mathbb{V})$ is a vector space over \mathbb{F} with respect to addition of linear maps and scalar multiplication.

recall that if $T, S : \mathbb{U} \longrightarrow \mathbb{V}$ are two linear maps and \mathcal{B} is a basis for \mathbb{U} , \mathcal{B}' is a basis for \mathbb{V} , then

$$\begin{aligned} [T + S]_{\mathcal{B}'}^{\mathcal{B}'} &= [T + S]_{\mathcal{B}'}^{\mathcal{B}'} \\ [\alpha T]_{\mathcal{B}'}^{\mathcal{B}'} &= \alpha [T]_{\mathcal{B}'}^{\mathcal{B}'} \end{aligned}$$

This means that the map that sends T to $[T]_{\mathcal{B}'}^{\mathcal{B}'}$ is linear.

The map $\varphi : Hom(\mathbb{U}, \mathbb{V}) \longrightarrow M_{m \times n}(\mathbb{F})$ such that $\varphi(T) = [T]_{\mathcal{B}'}^{\mathcal{B}'}$ is a linear map!

We will show that φ is one-to-one and onto, and conclude that

$$Hom(\mathbb{U}, \mathbb{V}) \cong M_{m \times n}(\mathbb{F})$$

or equivalently :

$$dim(Hom(\mathbb{U}, \mathbb{V})) = dim(\mathbb{U})dim(\mathbb{V})$$

φ is one to one : We will show that $Ker(\varphi) = \{0\}$ (0 map).

Let $T \in Ker(\varphi)$, so $\varphi(T) = 0$ i.e. $[T]_{\mathcal{B}'}^{\mathcal{B}'} = 0$ (0 matrix) recall :

$$\begin{aligned} T(u_1) &= 0v_1 + 0v_2 + \cdots + 0v_m = 0_{\mathbb{V}} \\ T(u_2) &= 0v_1 + 0v_2 + \cdots + 0v_m = 0_{\mathbb{V}} \\ &\vdots \\ &\vdots \\ T(u_n) &= 0v_1 + 0v_2 + \cdots + 0v_m = 0_{\mathbb{V}} \end{aligned}$$

Thus

$$T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) = 0_{\mathbb{V}}, \forall u \in \mathbb{U}$$

φ is onto. Let $A_{m \times n}(\mathbb{F})$ be a matrix. let's find a $T \in \text{Hom}(\mathbb{U}, \mathbb{V})$ s.t

$$\varphi(T) = [T]_{\mathcal{B}}^{\mathcal{B}'} = A = (a_{ij}),$$

define :

$$T(u_1) = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_m$$

$$T(u_2) = a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_m$$

$$\vdots$$

$$\vdots$$

$$T(u_n) = a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{nn}v_m$$

and then indeed T gives rise to a linear map by

$$T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n)$$

whose matrix representation is A .

Exercise 16. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ be a linear transformation such that $\dim \ker T = 2$. What is the dimension of $\text{Im } T$? Can T be surjective? Can T be injective?

Solution 16. By the rank-nullity theorem,

$$\dim \ker T + \dim \text{Im } T = \dim \mathbb{R}^5 = 5.$$

Thus

$$\dim \text{Im } T = 5 - 2 = 3.$$

Since the codomain \mathbb{R}^4 has dimension 4, the image is a 3-dimensional subspace, so T is not surjective. It is not injective because $\ker T \neq \{0\}$ (dimension 2).

Exercise 17. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear map defined as

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x + 2y + 3z \\ y + z \end{pmatrix}$$

1. Compute a basis of $\ker(f)$ and a basis of $\text{Im}(f)$,
2. Determine if these two subspaces are in direct sum..

Solution 17. 1.

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x + 2y + 3z \\ y + z \end{pmatrix}.$$

Kernel :

$$\ker(f) = \{u \in \mathbb{R}^3, f(u) = o_{\mathbb{R}^3}\}$$

Solve

$$\begin{cases} x - y = 0 \\ x + 2y + 3z = 0 \\ y + z = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ x + 2y + 3z = 0 \\ y = -z \end{cases} \Rightarrow \begin{cases} 0 \cdot x = 0 \\ y = x \\ z = -x \end{cases} .$$

Automatically. So free variable x . Then $y = x$, $z = -x$.

Kernel basis :

$$\ker(f) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \neq 0_{\mathbb{R}^3}$$

So. $\dim \ker(f) = 1$. **Image :**

$$\text{Im}(f) = \{f(u) \in \mathbb{R}^3, u \in \mathbb{R}^3\}$$

Using rank's theorem

$$\dim(\mathbb{R}^3) = \dim(\ker(f)) + \dim(\text{Im}(f)) \Rightarrow \dim(\text{Im}(f)) = 3 - 1 = 2$$

We have

$$f(u) = \begin{pmatrix} x - y \\ x + 2y + 3z \\ y + z \end{pmatrix} = \begin{pmatrix} x - y \\ (x - y) + 3(y + z) \\ y + z \end{pmatrix} = (x - y) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (y + z) \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

So

$$\text{Im}(f) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}. \quad \text{These two are independent.}$$

Basis : $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$. So $\dim(\text{Im}(f)) = 2$.

2. We compute $\ker(f) \cap \text{Im}(f)$

$$\begin{aligned} w \in \ker(f) \cap \text{Im}(f) &\Leftrightarrow w = \alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad w = \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} \alpha = \beta \\ \alpha = \beta + 3\gamma \\ -\alpha = \gamma \end{cases} \Leftrightarrow \alpha = \beta = \gamma = 0 \end{aligned}$$

Thus $\ker(f) \cap \text{Im}(f) = \{0_{\mathbb{R}^3}\}$. So the two subspaces $\ker(f)$ and $\text{Im}(f)$ are in direct sum.

$$\mathbb{R}^3 = \ker(f) \oplus \text{Im}(f)$$

Exercise 18. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear map such that

$$f \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad f \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

(-) Compute $f \begin{pmatrix} 8 \\ 14 \\ 20 \end{pmatrix}$

Solution 18.

(-)

$$f \begin{pmatrix} 8 \\ 14 \\ 20 \end{pmatrix} = f \left(4 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) + f \left(2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) = 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

Exercise 19. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map such that

$$f \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad f \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \quad \text{and} \quad f \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

(a) Compute $f \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) Compute the dimensions of $\ker f$ and $\text{Im} f$.

Solution 19.

(a) We have

$$f \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad \text{and} \quad f \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

and

$$f \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \Rightarrow f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

So, the matrix of f with respect to standard bases is

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 1 & 3 \\ 0 & 3 & 4 \end{pmatrix}.$$

Thus,

$$f \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 1 & 3 \\ 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$$

(b) **Kernel and Image :**

$$\ker(f) = \{u \in \mathbb{R}^3, f(u) = Au = o_{\mathbb{R}^3}\}$$

$$\text{Let } u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

$$\begin{aligned} u \in \ker(f) &\Leftrightarrow Au = o_{\mathbb{R}^3} \\ &\Leftrightarrow \begin{cases} -x + 2y + 2z = 0 \\ 2x + y + 3z = 0 \\ 3y + 4z = 0 \end{cases} \Leftrightarrow u = o_{\mathbb{R}^3} \end{aligned}$$

Therefore

$$\ker(f) = \{o_{\mathbb{R}^3}\} \Rightarrow \dim(\ker(f)) = 0$$

Using Rank's theorem

$$\dim(\mathbb{R}^3) = \dim(\ker(f)) + \dim(\text{Im}(f))$$

we get

$$\dim(\text{Im}(f)) = 3 - 0 = 3$$

Exercise 20. Let $T : \mathbb{Q}[x]_{\leq 3} \rightarrow \mathbb{Q}^3$ be a linear map such that

$$f(ax^3 + bx^2 + cx + d) = \begin{pmatrix} a + b \\ c + d \\ a + b + c + d \end{pmatrix}$$

(-) Compute a basis of $\ker(T)$ and a basis of $\text{Im}(T)$

Solution 20.

(-) **Kernel and Image :**

$$\ker(f) = \{P \in \mathbb{Q}[x]_{\leq 3}, f(p) = o_{\mathbb{Q}^3}\}$$

$$\text{Let } p(x) = (ax^3 + bx^2 + cx + d) \in \mathbb{Q}[x]_{\leq 3}$$

$$p \in \ker(f) \Leftrightarrow f(p) = o_{\mathbb{Q}^3}$$

$$\Leftrightarrow \begin{cases} a + b = 0 \\ c + d = 0 \\ a + b + c + d = 0 \end{cases} \Leftrightarrow \begin{cases} a = -b \\ c = -d \\ 0 = 0 \end{cases}$$

Thus

$$p \in \ker(f) \Leftrightarrow p(x) = ax^3 - ax^2 + cx - c = a(x^3 - x^2) + c(x - 1)$$

So,

$$\ker(f) = \text{span} \{x^3 - x^2, x - 1\}, \quad \text{These two vectors are independent.}$$

Therefore $\{x^3 - x^2, x - 1\}$, basis of $\ker(f)$ and $\dim(\ker(f)) = 2$.

Using rank's theorem, we get

$$\dim(\mathbb{Q}[x]_{\leq 3}) = \dim(\ker(f)) + \dim(\text{Im}(f)) \Rightarrow \dim(\text{Im}(f)) = 4 - 2 = 2$$

We have, for all vector $w \in \mathbb{Q}^3$.

$$\begin{aligned} w \in \text{Im}(f) &\Leftrightarrow w = \begin{pmatrix} a+b \\ c+d \\ a+b+c+d \end{pmatrix} \\ &\Leftrightarrow w = (a+b) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (c+d) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

So,

$$\text{Im}(f) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad \text{These two vectors are independent.}$$

$$\text{Thus } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ basis of } \text{Im}(f)$$

Exercise 21. Let $L : \mathbb{Q}^4 \rightarrow \mathbb{Q}[x]_{\leq 3}$ the linear map defined by

$$L \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a+b)x^3 + (c+d)x^2 + 2cx + 2d$$

(-) Compute a basis of $\ker(L)$ and a basis of $\text{Im}(L)$

Solution 21.

(-) **Kernel and Image :**

$$\ker(L) = \left\{ u \in \mathbb{Q}^4, L(u) = 0_{\mathbb{Q}[x]_{\leq 3}} \right\}$$

$$\text{Let } u = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{Q}^4$$

$$\begin{aligned} u \in \ker(L) &\Leftrightarrow L(u) = 0_{\mathbb{Q}[x]_{\leq 3}} \\ &\Leftrightarrow (a+b)x^3 + (c+d)x^2 + 2cx + 2d = 0_{\mathbb{Q}[x]_{\leq 3}} \\ &\Leftrightarrow \begin{cases} a+b=0 \\ c+d=0 \\ 2c=0 \\ 2d=0 \end{cases} \Leftrightarrow \begin{cases} a=-b \\ c=-d \\ c=0 \\ d=0 \end{cases} \end{aligned}$$

Thus

$$u \in \ker(L) \Leftrightarrow p(x) = 0_{\mathbb{Q}[x]_{\leq 3}}$$

So,

$$\ker(L) = \left\{ 0_{\mathbb{Q}[x]_{\leq 3}} \right\} \Leftrightarrow \dim(\ker(L)) = 0$$

Using rank's theorem, we get

$$\dim(\mathbb{Q}^4) = \dim(\ker(f)) + \dim(\text{Im}(f)) \Rightarrow \dim(\text{Im}(f)) = 4 - 0 = 4$$

We have $\text{Im}(L) = \mathbb{Q}[x]_{\leq 3}$ So, we can take $\{x^3, x^2, x, 1\}$ basis of $\text{Im}(L)$.

Matrices and Determinants

3.1 Matrices

Definition 3.1.1. A matrix over a field \mathbb{F} is a chart of elements of \mathbb{F} arranged in rows and columns.

Example 44.

$A = \begin{pmatrix} 0 & i & \pi \\ 4 & \sqrt{2} & -2i \end{pmatrix}$ is a matrix over a field \mathbb{C} , $B = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, is a matrix over a field \mathbb{R} .

In general

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

a_{ij} is the entry in row i and column j

More notation : $A = (a_{ij})$, $A \in \mathcal{M}_{mn}(\mathbb{F})$. If $n = m$ then $A \in \mathcal{M}_n(\mathbb{F})$.

Remark 3.1.1. In this course a field \mathbb{F} will be \mathbb{R} or \mathbb{C} .

Terminology

1. **Size of a matrix :** \mathcal{M}_{mn} " m " means the number of rows and " n " the number of columns.
2. **Square matrix :** if $m = n$ and $A \in \mathcal{M}_n$, A is called a **square matrix**.
3. **Main diagonal matrix :** All elements a_{ii} .
4. **Zero matrix :** a matrix with $a_{ij} = 0$, $\forall i, j$
5. **Identity matrix :** Identity matrix of size n denoted I_n is a square matrix where all

diagonal entries are $a_{ii} = 1$ and all off-diagonal entries are $a_{ij} = 0$, ($i \neq j$).

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Example 45.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. **Diagonal matrix :** Diagonal matrix of size n is a square matrix whose only nonzero entries are on the main diagonal. (every $a_{ij} = 0$, ($i \neq j$))

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

Example 46.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

7. **Scalar matrix :** is a square matrix where all diagonal entries are $a_{ii} = c$, $\forall i$ and all off-diagonal entries are $a_{ij} = 0$, ($i \neq j$)

$$\begin{pmatrix} c & 0 & 0 & \cdots & 0 & 0 \\ 0 & c & 0 & \cdots & 0 & 0 \\ 0 & 0 & c & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & c \end{pmatrix}$$

8. **Transpose matrix :** The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^t whose rows are the columns of A . In other words, the a_{ij} entry of A^t is a_{ji} . In other words. If $A \in \mathcal{M}_{mn}$ the transpose matrix of A is a matrix noted A^T or A^t or tA s.t $A^t \in \mathcal{M}_{nm}$ i.e

$$A \in \mathcal{M}_{nm}(\mathbb{F}), \quad A = (a_{ij}) \Leftrightarrow A^t \in \mathcal{M}_{mn}(\mathbb{F}) \quad A^t = (a_{ji}).$$

Example 47.

$$A = \begin{pmatrix} 1 & -5 \\ 1/3 & 2 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 1 & 1/3 \\ -5 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 & -4 \\ 0 & \pi & 7 \\ -1 & 2 & \sqrt{2} \end{pmatrix} \Rightarrow B^t = \begin{pmatrix} 1 & 0 & -1 \\ 3 & \pi & 2 \\ -4 & 7 & \sqrt{2} \end{pmatrix}$$

9. **Symmetric matrix** : $A^t = A$ i.e. $a_{ij} = a_{ji}$

Example 48. *The matrices A and B are Symmetric*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, A^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \Rightarrow A = A^t$$

$$B = \begin{pmatrix} 1 & 3 & -4 \\ 3 & \pi & 7 \\ -4 & 7 & \sqrt{2} \end{pmatrix}, B^t = \begin{pmatrix} 1 & 3 & -4 \\ 3 & \pi & 7 \\ -4 & 7 & \sqrt{2} \end{pmatrix} \Rightarrow B = B^t$$

10. **Skew-Symmetric matrix** : $A^t = -A$ i.e. $a_{ij} = -a_{ji}$

Example 49.

$$A = \begin{pmatrix} 0 & 3 & 4 \\ -3 & 0 & -7 \\ -4 & 7 & 0 \end{pmatrix}$$

11. **Upper triangular matrix** : Is a square matrix where $a_{ij} = 0$, for $i > j$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & \ddots & \cdots & a_{11} \\ 0 & 0 & 0 & \ddots & \ddots & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

Example 50.

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 3 \\ 0 & 6 & 4 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 0 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

12. **Lower triangular matrix** : Is a square matrix where $a_{ij} = 0$, for $i < j$

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n(n-1)} & a_{nn} \end{pmatrix}$$

Example 51.

$$\left(\begin{array}{cc} 1 & 0 \\ 4 & 3 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 0 & 1 & -2 \end{array} \right), \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -\sqrt{2} & 2 & 0 & 0 \\ -1 & 3 & 4 & 0 \\ -1 & 4 & \pi & -3 \end{array} \right)$$

13. **Row vector** : is matrix with one row

$$\left(a_1 a_2 a_3 \cdots a_n \right)$$

Example 52.

$$\left(1, 0 \right), \left(-1, 0, -\sqrt{3} + 1 \right), \left(1, 2, -\sqrt{3}, 0 \right)$$

14. **Column vector** : is matrix with one column

$$\left(\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{array} \right)$$

Example 53.

$$\left(\begin{array}{c} 1 \\ 3 \end{array} \right), \left(\begin{array}{c} 1 \\ -6 \\ 7 \end{array} \right), \left(\begin{array}{c} 1 \\ 0 \\ 2 \\ 3 \end{array} \right)$$

15. **Scalar is matrix** : if $a \in \mathbb{F}$ is a scalar then $a \in \mathcal{M}_1(\mathbb{F})$

Example 54.

$$1, \sqrt{2}, \pi, \dots$$

16. **Trace of matrix** : The trace of the square matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is a real number

$$tr(A) = \sum_{i=1}^n a_{ii}$$

Example 55. $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$, $tr(A) = 1 + (-2) = -1$,

$$B = \begin{pmatrix} 1 & 2 & -4 \\ 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}, \quad tr(B) = 1 + (-2) + 1 = 0$$

3.1.1 Operations on Matrices

Let $A, B, C \in \mathcal{M}_{nm}(\mathbb{F})$ and $\alpha \in \mathbb{F}$ s.t :

$$A = (a_{ij}), \quad B = (b_{ij}), \quad C = (c_{ij})$$

1. **Equality** : $A = B$ if they have exactly the same size and exactly the same entries.

$$A = B \Leftrightarrow a_{ij} = b_{ij}$$

Example 56.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha & 0 \\ 1 & \beta & 0 \\ \gamma & 1 & -2 \end{pmatrix} \Leftrightarrow \begin{cases} \alpha = 0 \\ \beta = 6 \\ \gamma = 0 \end{cases}$$

2. **Addition** : element-wise

$$A \pm B = (a_{ij} \pm b_{ij})$$

Example 57.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 0 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 0 \\ -3 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \\ 2 & 7 & 0 \\ -3 & 2 & -4 \end{pmatrix}$$

3. **Scalar times a matrix** : entry-wise

$$\alpha A = (\alpha a_{ij})$$

Example 58.

$$3 \begin{pmatrix} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 18 & 0 \\ 0 & 3 & -6 \end{pmatrix}, \quad \alpha \begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 2\alpha & -2\alpha \\ -3\alpha & \alpha \end{pmatrix}$$

Properties 3.1.1. Let $A, B, C \in \mathcal{M}_{nm}(\mathbb{F})$ and $\alpha, \beta \in \mathbb{F}$

1. $A + B = B + A$ *Commutative Law of Addition*
2. $(A + B) + C = A + (B + C)$ *Associative Law of Addition*
3. $A + 0 = 0 + A = A$ (0 is a zero matrix) *Existence of an Additive Identity.*
4. $A + (-A) = (-A) + A = 0$ *Existence of an Additive Inverse*
5. $\alpha(A + B) = \alpha A + \alpha B$ *Distributive Law over Matrix Addition*
6. $(\alpha + \beta)A = \alpha A + \beta A$ *Distributive Law over Scalar Addition*
7. $(\alpha\beta)A = \alpha(\beta A)$ *Associative Law for Scalar Multiplication*
8. $1_{\mathbb{F}}A = A$ *Rule for Multiplication by $1_{\mathbb{F}}$*
9. $(A \pm B)^t = A^t \pm B^t$
10. $(\alpha A)^t = \alpha A^t$
11. $(A^t)^t = A$

Corollary 3.1.1. The set $\mathcal{M}_{nm}(\mathbb{F})$ is a vector space over a field \mathbb{F}

Example 59. $\mathcal{M}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{R} \right\}$ is vector space over a field \mathbb{R} .

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. So

$$A = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We can easily prove that the set $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of the vector space $\mathcal{M}_2(\mathbb{R})$ and $\dim(\mathcal{M}_2(\mathbb{R})) = 4$

Exercise 22. Which of the following subsets of the vector space $M_{2 \times 2}(\mathbb{R})$ (real 2×2 matrices) are subspaces?

1. $W_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}$ (traceless matrices).
2. $W_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$ (matrices with determinant 1).
3. $W_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ (integer entries).
4. $W_4 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = 0 \right\}$.

Solution 22.

1. W_1 is a subspace. Check :

- Zero matrix has $0 + 0 = 0$.
- If $A, B \in W_1$, then $\text{tr}(A) = 0$, $\text{tr}(B) = 0$. Then

$$\text{tr}(A + B) = 0 + 0 = 0.$$

- For scalar α ,

$$\text{tr}(\alpha A) = \alpha \text{tr}(A) = 0.$$

So W_1 is a subspace.

2. W_2 is not a subspace. The zero matrix has determinant 0, not 1, so it fails the first condition.
3. W_3 is not a subspace because scalar multiplication can lead to non-integer entries. For example, take matrix with integer entries and multiply by $\frac{1}{2}$; the result may not have integer entries. Closure under scalar multiplication fails.
4. W_4 is a subspace. Conditions : all matrices with top-left entry zero. Check :
 - Zero matrix has $a = 0$.
 - If two matrices have $a = 0$ and $a' = 0$, their sum also has top-left entry $0 + 0 = 0$.
 - Scalar multiple : if $a = 0$, then $\alpha a = 0$.

Thus W_4 is a subspace.

Claim 3.1.

1. In general

$$\dim(\mathcal{M}_{nm}(\mathbb{R})) = n \times m \quad \text{and} \quad \dim(\mathcal{M}_n(\mathbb{R})) = n^2$$

2. We can prove that

$$\mathcal{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

Example 60. Let T be a map such that

$$T : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

T is isomorphism so, we get $\mathcal{M}_2(\mathbb{R}) \cong \mathbb{R}^4$

Matrix multiplication

Definition 3.1.2. The matrix-multiplicative product of the matrix $A \in \mathcal{M}_{nm}(\mathbb{F})$ and the matrix $B \in \mathcal{M}_{ml}(\mathbb{F})$ is the matrix $C \in \mathcal{M}_{nl}(\mathbb{F})$ where

$$c_{ij} = \sum_{k=1}^m a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

Another way to write this is

$$(AB)_{ij} = (a_{i1}, a_{i2}, \dots, a_{im}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

$$\underbrace{\begin{pmatrix} \vdots \\ \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{im} \\ \vdots \\ \vdots \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{1j} \\ b_{2j} \\ \cdots & b_{3j} & \cdots \\ \vdots \\ b_{mj} \end{pmatrix}}_B = \underbrace{\begin{pmatrix} \sum_{k=1}^m a_{i1}b_{1j} \\ \sum_{k=1}^m a_{i2}b_{2j} \\ \cdots & \sum_{k=1}^m a_{ik}b_{kj} & \cdots \\ \vdots \\ \sum_{k=1}^m a_{im}b_{mj} \end{pmatrix}}_C$$

so that the i, j -th entry of the product is the dot product of the i -th row of the first matrix with the j -th column of the second.

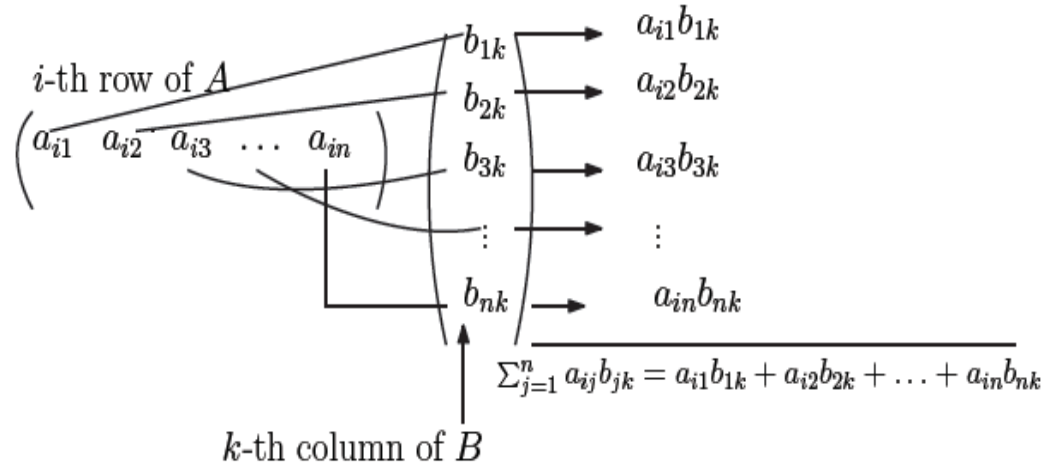
Example 61.

$$\begin{pmatrix} 2 & 0 \\ 4 & 6 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 0 \times 5 & 2 \times 3 + 0 \times 7 \\ 4 \times 1 + 6 \times 5 & 4 \times 3 + 6 \times 7 \\ 8 \times 1 + 2 \times 5 & 8 \times 3 + 2 \times 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 34 & 54 \\ 18 & 38 \end{pmatrix}$$

Some products are not defined, such as the product of a 2×3 matrix with a 2×2 , because the number of columns in the first matrix must equal the number of rows in the second. But the product of two $n \times n$ matrices is always defined.

Example 62.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 \times (-1) + 2 \times 2 & 1 \times 0 + 2 \times (-2) \\ 3 \times (-1) + 4 \times 2 & 3 \times 0 + 4 \times (-2) \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 5 & -8 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & 9 & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix}$$

Properties of matrix multiplication

Properties 3.1.2. Let $A \in \mathcal{M}_{mn}$, $B \in \mathcal{M}_{nl}$, $D \in \mathcal{M}_{nl}$ and $C \in \mathcal{M}_{lg}$ e matrices, then

1. $(AB)D = A(BD)$
2. $A(B + C) = AB + AC$
3. $(D + B)C = DC + BC$
4. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
5. $AI_n = I_m A = A$
6. $A0 = 0A = 0$ for which the products are defined.
7. $(AB)^t = B^t A^t$

Proof :

1.

$$\begin{aligned}
 (AB)C &= \left(\sum_{j=1}^n a_{ij}b_{jk} \right) c_{kl} \\
 &= \left(\sum_{k=1}^p \left[\sum_{j=1}^n a_{ij}b_{jk} \right] c_{kl} \right) \\
 &= \left(\sum_{j=1}^n a_{ij} \sum_{k=1}^p (b_{jk}c_{kl}) \right) \\
 &= a_{ij} \left(\sum_{k=1}^p (b_{jk}c_{kl}) \right) \\
 &= A(BC)
 \end{aligned}$$

2. The proof follows from the fact that

$$A(B + C) = \sum_{j=1}^n a_{ij}(b_{jk} + c_{jk}) = \sum_{j=1}^n a_{ij}b_{jk} + \sum_{j=1}^n a_{ij}c_{jk} = AB + AC$$

3. In the same way as propriety 2

4.

$$\alpha(AB) = \alpha\left(\sum_{j=1}^n a_{ij}b_{jk}\right) = \left(\sum_{j=1}^n (\alpha a_{ij})b_{jk}\right) = (\alpha A)B = \left(\sum_{j=1}^n a_{ij}(\alpha b_{jk})\right) = A(\alpha B)$$

5. Very easy to proof

6. Very easy to proof

7.

$$(AB)^t = \left(\sum_{j=1}^n a_{ij}b_{jk}\right)^t$$

Remarks 3.1.1.

— In general, matrix multiplication is not commutative.

— $\forall A \in \mathcal{M}_n(\mathbb{R}), AI_n = I_n A$

Exercise 23. Let

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad D = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}.$$

Compute the following, if possible :

1. $A + B$
2. $2A - 3B$
3. AB and BA
4. CD and DC
5. A^T and C^T
6. $\text{tr}(A)$ and $\text{tr}(B)$

Solution 23.

1.

$$A + B = \begin{pmatrix} 2+1 & -1+4 \\ 3-2 & 0+5 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}.$$

2.

$$2A = \begin{pmatrix} 4 & -2 \\ 6 & 0 \end{pmatrix}, \quad 3B = \begin{pmatrix} 3 & 12 \\ -6 & 15 \end{pmatrix},$$

so

$$2A - 3B = \begin{pmatrix} 4-3 & -2-12 \\ 6+6 & 0-15 \end{pmatrix} = \begin{pmatrix} 1 & -14 \\ 12 & -15 \end{pmatrix}.$$

3.

$$AB = \begin{pmatrix} 2 \cdot 1 + (-1)(-2) & 2 \cdot 4 + (-1)5 \\ 3 \cdot 1 + 0(-2) & 3 \cdot 4 + 0 \cdot 5 \end{pmatrix} = \begin{pmatrix} 2 + 2 & 8 - 5 \\ 3 + 0 & 12 + 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 12 \end{pmatrix}.$$

$$BA = \begin{pmatrix} 1 \cdot 2 + 4 \cdot 3 & 1(-1) + 4 \cdot 0 \\ -2 \cdot 2 + 5 \cdot 3 & -2(-1) + 5 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2 + 12 & -1 + 0 \\ -4 + 15 & 2 + 0 \end{pmatrix} = \begin{pmatrix} 14 & -1 \\ 11 & 2 \end{pmatrix}.$$

4. C is 2×3 , D is 3×2 , so CD is 2×2 :

$$CD = \begin{pmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{pmatrix}$$

$$= \begin{pmatrix} 7 + 18 + 33 & 8 + 20 + 36 \\ 28 + 45 + 66 & 32 + 50 + 72 \end{pmatrix}$$

$$= \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}.$$

 DC is 3×3 :

$$DC = \begin{pmatrix} 7 \cdot 1 + 8 \cdot 4 & 7 \cdot 2 + 8 \cdot 5 & 7 \cdot 3 + 8 \cdot 6 \\ 9 \cdot 1 + 10 \cdot 4 & 9 \cdot 2 + 10 \cdot 5 & 9 \cdot 3 + 10 \cdot 6 \\ 11 \cdot 1 + 12 \cdot 4 & 11 \cdot 2 + 12 \cdot 5 & 11 \cdot 3 + 12 \cdot 6 \end{pmatrix}$$

$$= \begin{pmatrix} 7 + 32 & 14 + 40 & 21 + 48 \\ 9 + 40 & 18 + 50 & 27 + 60 \\ 11 + 48 & 22 + 60 & 33 + 72 \end{pmatrix}$$

$$= \begin{pmatrix} 39 & 54 & 69 \\ 49 & 68 & 87 \\ 59 & 82 & 105 \end{pmatrix}.$$

5.

$$A^t = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad C^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

6.

$$\text{tr}(A) = 2 + 0 = 2, \quad \text{tr}(B) = 1 + 5 = 6.$$

Vector Multiplied by a Matrix

Definition 3.1.3. Let $A \in \mathcal{M}_{mn}$ be a matrix and let v be an $n \times 1$ matrix (vector column) given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Then the product Av is the $m \times 1$ column vector which equals the following linear combi-

nation of the columns of A :

$$Av = \begin{pmatrix} \sum_{i=1}^m a_{i1}v_1 \\ \sum_{i=1}^m a_{i2}v_2 \\ \vdots \\ \sum_{i=1}^m a_{in}v_n \end{pmatrix}$$

If we write the columns of A in terms of their entries, they are of the form

$$A = \left(\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right), \text{ we put } A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Then, we can write the product Av as

$$Av = \sum_{j=1}^n v_j A_j = v_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + v_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Example 63. Compute the product Av for

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 1 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Solution : We will use **Definition(3.1.3)** to compute the product. Therefore, we compute the product Av as

$$Av = 1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 5 \end{pmatrix}$$

Powers of Square Matrices

Definition 3.1.4. Let $A \in \mathcal{M}_n(\mathbb{R})$ be any square matrix. Then the (nonnegative) powers of A are given by

$$A^0 = I_n, \quad A^1 = A, \quad A^2 = AA, \quad \dots \quad A^p = A^{p-1}A$$

Example 64. Suppose that $A = \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$. Then

$$A^2 = \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}^2 = \begin{pmatrix} 0 & 5 \\ -20 & 5 \end{pmatrix}, \quad A^3 = A^2A = \begin{pmatrix} 0 & 5 \\ -20 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -20 & 15 \\ -60 & -5 \end{pmatrix}$$

Remark 3.1.2.

$$\forall p \in \mathbb{N}, (I_n)^p = I_n$$

Example 65. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Find A^2 , A^3 and A^4 .

Solution : We get

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, A^3 = A^2A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}, A^4 = A^3A = \begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

We can easily conclude that

$$\forall p \in \mathbb{N}, A^p = \begin{pmatrix} 1 & 0 & 2^p - 1 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 2^p \end{pmatrix}$$

Using proof by recurrence

Remark 3.1.3. As multiplication is not commutative :

$$(A + B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$$

Example 66. Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$. We have

$$(A + B)^2 = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^2 = \begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix}$$

But

$$A^2 + 2AB + B^2 = \begin{pmatrix} 0 & -3 \\ 0 & 4 \end{pmatrix} + 2 \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -6 & 13 \end{pmatrix}$$

Proposition 3.1.1. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_n(\mathbb{R})$ be two square matrices,

$$\text{if } AB = BA, \text{ then } (A + B)^2 = A^2 + 2AB + B^2$$

and

$$(A + B)^p = \sum_{k=0}^p \binom{p}{k} A^{p-k} B^k$$

$$\text{where } : \binom{p}{k} = C_k^p = \frac{p!}{k!(p-k)!}$$

Example 67. let's calculate A^p , where $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

We pose $N = A - I_4$, we get

$$N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N^2 = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N^3 = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } N^4 = 0_4$$

The matrix N is said to be **nilpotent**.

As $A = I_4 + N$ and the matrices N and I_4 commute then

$$\begin{aligned} A^p &= (I_4 + N)^p = \sum_{k=0}^p \binom{p}{k} I_4^{p-k} N^k \\ &= I_4 + pN + \frac{p(p-1)}{2!} N^2 + \frac{p(p-1)(p-2)}{3!} N^3, \text{ because } N^k = 0_4, \forall k \geq 4 \end{aligned}$$

Therefore

$$A^p = \begin{pmatrix} 1 & p & p^2 & p(p^2 - p + 1) \\ 0 & 1 & 2p & p(3p - 2) \\ 0 & 0 & 1 & 3p \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 24. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix}$, calculate A^p and B^p for all natural p . Prove that $AB = BA$ and find $(A + B)^p$

Solution 24. 1.

$$A = 2I_3 \quad \text{so} \quad A^p = 2^p I_3.$$

2. B is nilpotent :

$$B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad B^3 = 0. \quad \text{Thus for } p \geq 3, B^p = 0.$$

3.

$$AB = 2I_3 B = 2B = B(2I_3) = BA.$$

4. Since A and B commute,

$$(A + B)^p = \sum_{k=0}^p \binom{p}{k} A^{p-k} B^k. \quad \text{For } p \geq 3, \text{ terms with } k \geq 3 \text{ vanish.}$$

So :

$$(A + B)^p = \sum_{k=0}^2 \binom{p}{k} (2I_3)^{p-k} B^k = 2^p I_3 + p \cdot 2^{p-1} B + \frac{p(p-1)}{2} 2^{p-2} B^2.$$

For $p = 1, 2$ the formula still holds (with $B^2 = 0$ when $p = 1$).

Substituting a matrix into a polynomial

Example 68.

Let, $p(x) = 3x^3 - x + 2$ and $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. We can write

$$\begin{aligned} p(A) &= 3A^3 - 5A + 2I_2 \\ &= 3 \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}^3 - 5 \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -4 \\ 0 & 4 \end{pmatrix} = 4 \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Inverse matrices

Definition 3.1.5. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix, if there exists a matrix $B \in \mathcal{M}_n(\mathbb{R})$ such that $AB = BA = I_n$ then we say that A is invertible and we note $B = A^{-1}$. Invertible matrices are also called nonsingular. If a matrix has no inverse then it is called a noninvertible or singular matrix.

$$A \in \mathcal{M}_n(\mathbb{R}) \text{ Invertible} \Leftrightarrow \exists B \in \mathcal{M}_n(\mathbb{R}), AB = BA = I_n$$

If A is invertible

$$\forall p \in \mathbb{N}, A^{-p} = (A^{-1})^p = \underbrace{A^{-1}A^{-1} \dots A^{-1}}_{p \text{ factors}}$$

The set of invertible matrices is called **general linear group** $GL_n(\mathbb{F})$.

$$GL_n(\mathbb{F}) = \{A \in \mathcal{M}_n(\mathbb{R}), A \text{ is invertible}\}$$

We'll discuss how and when it is possible to calculate A^{-1} for a given square matrix A , however, we need to develop a few tools before we're ready for that problem. This much I can share now :

Example 69. Consider the problem of inverting a square matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. Let's

search a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AB = BA = I_2$.

First

$$AB = I_2 \Leftrightarrow \begin{pmatrix} a+2c & b+2d \\ 3c & 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \begin{cases} a+2c = 1 \\ b+2d = 0 \\ 3c = 0 \\ 3d = 1 \end{cases}$$

There is only one solution $B = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/3 \end{pmatrix}$.

To prove that $B = A^{-1}$, we must also prove the equality $BA = I_2$.

Thus the matrix A invertible and $A^{-1} = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/3 \end{pmatrix}$

Example 70. The matrix $A = \begin{pmatrix} 3 & 0 \\ 5 & 0 \end{pmatrix}$ is not invertible

$$BA = I_2 \Leftrightarrow \begin{pmatrix} 3a + 5b & 0 \\ 3c + 5d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \begin{cases} 3a + 5b = 1 \\ 3c + 5d = 0 \\ 0 = 0 \\ 0 = 1, \text{ Impossible} \end{cases}$$

Example 71.

- The identity matrix I_n is invertible and $I_n^{-1} = I_n$, by equality $I_n I_n = I_n$
- The zero matrix I_n is not invertible and $I_n^{-1} = I_n$, because $B0_n = 0_n \neq I_n$

Proposition 3.1.2. If the matrix $A \in \mathcal{M}_n(\mathbb{R})$ is invertible, then its inverse is unique

Proof : Suppose that there exist two different inverses B_1 and B_2 of A . As matrix multiplication is associative, then

$$B_2 = B_2 I_n = B_2 \underbrace{(AB_1)}_{I_n} = \underbrace{(B_2 A)}_{I_n} B_1 = B_1$$

Contradiction, thus the inverse is unique.

Proposition 3.1.3. If the matrices $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_n(\mathbb{R})$ are invertible, then :

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{-1} = A$
- $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}, \quad \alpha \neq 0$
- $(A^t)^{-1} = (A^{-1})^t$
- $(A^k)^{-1} = (A^{-1})^k, \quad k \in \mathbb{N}$

Proof :

— $(AB)^{-1} = B^{-1}A^{-1}$? We have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I_n)B = B^{-1}B = I_n$$

and

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$$

Therefore

$$(AB)^{-1} = B^{-1}A^{-1}$$

— $(A^{-1})^{-1} = A$? We have

$$\begin{aligned} I_n^{-1} = I_n &\Leftrightarrow (AA^{-1})^{-1} = I_n \\ &\Leftrightarrow (A^{-1})^{-1}A^{-1} = I_n \end{aligned}$$

and

$$\begin{aligned} I_n^{-1} = I_n &\Leftrightarrow (A^{-1}A)^{-1} = I_n \\ &\Leftrightarrow A^{-1}(A^{-1})^{-1} = I_n \end{aligned}$$

Thus $(A^{-1})^{-1} = A$

— It is clear that

$$\left(\frac{1}{\alpha}A^{-1}\right)(\alpha A) = \frac{\alpha}{\alpha}A^{-1}A = I_n$$

$$(\alpha A)\left(\frac{1}{\alpha}A^{-1}\right) = \frac{\alpha}{\alpha}AA^{-1} = I_n$$

Therefore $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$

— By definition we have

$$(A^{-1})^t A^t = (AA^{-1})^t = (I_n)^t = I_n$$

$$A^t (A^{-1})^t = (A^{-1}A)^t = (I_n)^t = I_n$$

Therefore $(A^{-1})^t = (A^t)^{-1}$

— Let $k \in \mathbb{N}$, we have

$$(A^k)^{-1} = \underbrace{(AAA \cdots A)^{-1}}_{k \text{ factors}} = \underbrace{A^{-1}A^{-1}A^{-1} \cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k$$

Corollary 3.1.2. Analogously, if $A_1, A_2, \dots, A_{m-1}, A_m$ are invertible, then

$$(A_1 A_2 A_3 \dots A_{m-1} A_m)^{-1} = A_m^{-1} A_{m-1}^{-1} \dots A_2^{-1} A_1^{-1}$$

Remarks 3.1.2.

— In general $AC = BC$ does not imply $A = B$

— In general $AC = 0$ does not imply $A = 0$ or $B = 0$

Example 72. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

— It is clear that $A \neq B$ but $AC = BC = 0_2$

— We have $AC = 0_2$ but $A \neq 0_2$ and $C \neq 0_2$. Here A and C both are non-zero matrices but $AC = 0_2$. Thus, the product of two non-zero matrices may be zero.

Proposition 3.1.4. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_n(\mathbb{R})$ be two matrices and $C \in \mathcal{M}_n(\mathbb{R})$ be an invertible matrix, then

$$AC = BC \Rightarrow A = B$$

Proof : We have

$$\begin{aligned} AC = BC &\Rightarrow (AC)C^{-1} = (BC)C^{-1} \\ &\Rightarrow A(CC^{-1}) = B(CC^{-1}) \\ &\Rightarrow AI_n = BI_n \\ &\Rightarrow A = B \end{aligned}$$

Exercise 25.

1. Let $A = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. Calculate A^{-1} , B^{-1} , $(AB)^{-1}$, $(BA)^{-1}$ and A^{-2}

2. Find the inverse of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

3. Let $A = \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Calculate $2A - A^2$ without calculations, deduce A^{-1} .

Solution 25. 1. We get

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ -5 & 3 \end{pmatrix}, \quad B^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 3 & -3 \\ 7 & -5 \end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{6} \begin{pmatrix} -5 & 3 \\ -7 & 3 \end{pmatrix}, \quad (BA)^{-1} = \frac{1}{6} \begin{pmatrix} 0 & 1 \\ -6 & -2 \end{pmatrix}, \quad A^{-2} = \frac{1}{9} \begin{pmatrix} 1 & 0 \\ -20 & 9 \end{pmatrix}$$

2.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

3.

$$\begin{aligned} 2A - A^2 &= 2 \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \\ &= \begin{pmatrix} -2 & -4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} -3 & -4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \end{aligned}$$

Thus $2A - A^2 = I_3$, from this equality we can get

$$A(2I_3 - A) = (2I_3 - A)A = I_3$$

which means that $(2I_3 - A)$ is the inverse of A , so

$$A^{-1} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Claim 3.2. Consider the problem of inverting a square matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $ad - bc \neq 0$, then the inverse of A is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 73.

1.

$$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$$

2. Consider the rotation by angle θ in the plane can be represented by a matrix

$$\text{Rot}(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Calculate the inverse formula

$$\text{Rot}^{-1}(\theta) = \frac{1}{\cos^2(\theta) + \sin^2(\theta)} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \frac{1}{1} \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \text{Rot}(-\theta)$$

We observe the inverse matrix corresponds to a rotation by angle $-\theta$, $\text{Rot}^{-1}(\theta) = \text{Rot}(-\theta)$. Notice that $\text{Rot}(0) = I_2$ thus $\text{Rot}(\theta)\text{Rot}^{-1}(\theta) = \text{Rot}(0) = I_2$. Rotations are very special invertible matrices.

Definition 3.1.6. The symbol $\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$ is called the **Kronecker delta**.

Example 74. The identity matrix I_n is defined by $I_n = (\delta_{ij})_{1 \leq i, j \leq n}$

3.1.2 Elementary row operations

Elementary row operations are used in **Gaussian elimination** to reduce a matrix to row echelon form. They are also used in **Gauss–Jordan elimination** to further reduce the matrix to reduced row echelon form.

There are three types of elementary matrices, which correspond to three types of row operations (respectively, column operations) :

1. **Row switching** A row within the matrix can be switched with another row.

$$R_i \leftrightarrow R_j$$

2. **Row multiplication** Each element in a row can be multiplied by a non-zero constant. It is also known as scaling a row.

$$kR_i \rightarrow R_j, \quad \text{where } k \neq 0$$

3. **Row addition** A row can be replaced by the sum of that row and a multiple of another row.

$$R_i + kR_j \rightarrow R_i$$

3.1.3 Ring of Square Matrices

Theorem 3.1.1. The set of square matrices has a ring structure for usual addition and multiplication.

Proof :

1. $(\mathcal{M}_n(\mathbb{F}), +)$ is an abelian group
2. The usual multiplication is associative
3. Multiplication is distributive with respect to addition,
4. The additive identity is unique

Example 75. *The set of 2-by-2 square matrices with entries in a field \mathbb{R} is*

$$\mathcal{M}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{R} \right\}$$

With the operations of usual matrix addition and usual matrix multiplication, $\mathcal{M}_2(\mathbb{R})$ satisfies the ring axioms.

The element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity of the ring. If

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

while

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

this example shows that the ring is noncommutative.

3.1.4 Vector Space of Matrices

Theorem 3.1.2. *Let $\mathcal{M}_{nm}(\mathbb{F})$ be the set of all $m \times n$ -matrices with entries in a field \mathbb{F} , together with the usual operations of matrix addition and scalar multiplication. Then $\mathcal{M}_{nm}(\mathbb{F})$ is a vector space*

Proof :

• Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be matrices of the same size, where $a_{ij}, b_{ij}, c_{ij} \in \mathbb{F}$. Since \mathbb{F} is Field, the following properties hold.

— The commutative law of addition

$$A + B = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = B + A.$$

— The associative law of addition

$$(A + B) + C = (a_{ij} + b_{ij}) + (c_{ij}) = (a_{ij}) + (b_{ij} + c_{ij}) = A + (B + C).$$

— The existence of an additive unit

$$A + 0 = (a_{ij}) + (0_{ij}) = (a_{ij} + 0_{ij}) = (a_{ij}) = A.$$

— The existence of an additive inverse

$$A + (-A) = (a_{ij}) + (-a_{ij}) = (a_{ij} - a_{ij}) = 0.$$

• Let A and B be matrices of the same size, and let $k, l \in \mathbb{F}$ be scalars. Then the following properties hold.

— The distributive law over matrix addition

$$k(A + B) = k(a_{ij} + b_{ij}) = (ka_{ij}) + (kb_{ij}) = k(a_{ij}) + k(b_{ij}) = kA + kB.$$

— The distributive law over scalar addition

$$(k + l)A = (k + l)(a_{ij}) = k(a_{ij}) + l(a_{ij}) = kA + lA.$$

— The associative law for scalar multiplication

$$k(lA) = k(l(a_{ij})) = (kla_{ij}) = (kl)(a_{ij}) = (kl)A.$$

— The rule for multiplication by 1

$$1A = 1((a_{ij})) = (1a_{ij}) = (a_{ij}) = A.$$

We now examine an example of a set that does not satisfy all of the above axioms, and is therefore not a vector space.

Example 76. Let V denote the set of 2×3 -matrices. Let us define a non-standard addition in V by $A \oplus B = A$ for all matrices $A, B \in V$. Let scalar multiplication in V be the usual scalar multiplication of matrices. Show that V is not a vector space.

Solution : In order to show that V is not a vector space, it suffices to find one of the 8 axioms that is not satisfied. We will begin by examining the axioms for addition until one is found which does not hold. In fact, for this example, the very first axiom fails. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then $A \oplus B = A$ and $B \oplus A = B$. Since $A \neq B$, we have $A \oplus B \neq B \oplus A$ for these two matrices. Thus V is not a vector space.

3.1.5 Column space, Row space

Definition 3.1.7. The set of all vectors that can be obtained by adding up scalar multiples of the columns of a matrix is called its **column space**. Similarly the **row space** is the set of all row vectors obtained by adding up multiples of the rows of a matrix.

3.1.6 Matrix of a Linear map

Let \mathbb{U} and \mathbb{V} be two vector spaces over a field \mathbb{F} such that $\dim(\mathbb{U}) = n$ and $\dim(\mathbb{V}) = m$. We know that

$$\mathbb{U} \cong \mathbb{F}^n \text{ and } \mathbb{V} \cong \mathbb{F}^m$$

Theorem 3.1.3. Let $T : \mathbb{U} \rightarrow \mathbb{V}$ be a linear map. Then we can find a matrix A such that $T(v) = Av$, $v \in \mathbb{U}$. In this case, we say that T is **determined or induced** by the matrix A .

Here is why. Suppose $T : (\mathbb{U} \cong \mathbb{F}^n) \rightarrow (\mathbb{V} \cong \mathbb{F}^m)$ is a linear map and you want to find the matrix defined by this linear map as $\forall v \in \mathbb{F}^n$, $T(v) = Av$.

Note that

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + v_n \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^n v_i e_i$$

where e_i is the i^{th} column of I_n , that is the $n \times 1$ vector which has zeros in every slot but the i^{th} and a 1 in this slot.

Then since T is linear,

$$\begin{aligned} T(v) &= \sum_{i=1}^n v_i T(e_i) \\ &= \underbrace{\begin{pmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{pmatrix}}_A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} \\ &= A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} \end{aligned}$$

The desired matrix is obtained from constructing the i^{th} column as $T(e_i)$. Recall that the set $\{e_1, e_2, \dots, e_n\}$ is called the **standard basis** or **canonical basis** of \mathbb{F}^n . Therefore the matrix of T is found by applying T to the standard basis. We state this formally as the following theorem

Theorem 3.1.4. *Let $T : \mathbb{U} \rightarrow \mathbb{V}$ be a linear map. Then the matrix A satisfying $T(v) = Av$, $v \in \mathbb{U}$, is given by*

$$A = \begin{pmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{pmatrix},$$

where $\{e_1, e_2, \dots, e_n\}$ is the standard basis of \mathbb{F}^n and $T(e_i)$ is the i^{th} column of A . A will be a $m \times n$ matrix

Example 77. *Suppose T is a linear map, $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ where*

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Find the matrix A of T such that $T(v) = Av$ for all $v \in \mathbb{R}^3$.

Solution : In this case, A will be a 2×3 matrix, so we need to find $T(e_1)$, $T(e_2)$ and $T(e_3)$. Luckily, we have been given these values so we can fill in A as needed, using these vectors as the columns of A . Hence,

$$A = \begin{pmatrix} 1 & 9 & 1 \\ 2 & -3 & 1 \end{pmatrix}$$

Thus T is defined by

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 9 & 1 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 9y + z \\ 2x - 3y + z \end{pmatrix}$$

Exercise 26. Suppose T is a linear map, $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where

$$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Find the matrix A of T such that $T(v) = Av$ for all $v \in \mathbb{R}^2$.

Solution 26. To find this matrix, we need to determine the action of T on e_1 and e_2 . In **Example (77)**, we were given these resulting vectors. However, in this example, we have been given T of two different vectors. How can we find out the action of T on e_1 and e_2 ? In particular for e_1 , suppose there exist x and y such that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Then, since T is linear,

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = xT \begin{pmatrix} 1 \\ 1 \end{pmatrix} + yT \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Substituting in values, this sum becomes

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Leftrightarrow \begin{cases} x = z - 2 \\ y = -t \end{cases}$$

Exercise 27. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the linear map defined as

$$T(v) = Av$$

where A is a matrix given by $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

1. Determine $T(v)$
2. Find a basis of $U = \ker(T)$ and a basis of $W = \text{Im}(T)$
3. Find a set of cartesian equations for W , i.e. write W as the set of solutions of a

homogeneous linear system.

Solution 27.

1.

$$T \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + y + z + t \\ y + t \\ x + z \end{pmatrix}.$$

2. **Kernel :**

$$\ker(T) = \{u \in \mathbb{R}^4, T(u) = o_{\mathbb{R}^3}\}$$

solve

$$\begin{cases} x + y + z + t = 0 \\ y + t = 0 \\ x + z = 0 \end{cases} \Rightarrow \begin{cases} x + y + z + t = 0 \\ y = -t \\ x = -z \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ y = -t \\ x = -z \end{cases}.$$

Automatically. So free variables x, y . Then $z = -x, t = -y$.

Kernel basis :

$$\ker(T) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}, \quad \text{These two vectors are independent.}$$

So, $\dim \ker = 2$.

3. **Image :** columns of A :

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = c_1, \quad c_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_2.$$

So

$$\text{Im } T = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

These two are independent. Basis : $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. So $\dim(\text{Im}(T)) = 2$.

4. **Cartesian equations for W :** Let $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, so

$$u \in W \Leftrightarrow u = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x = \alpha + \beta \\ y = \beta \\ z = \alpha \end{cases} \Leftrightarrow x - y - z = 0$$

second method : Since $\dim(W) = 2$, W is a plane in \mathbb{R}^3 . Find normal vector by cross product : $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Thus equation : $x - y - z = 0$. So

$$W = \{(x, y, z) \in \mathbb{R}^3 : -x + y + z = 0\}.$$

Exercise 28. $R_{3\pi/2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a rotation by $3\pi/2$ find $R_{(3\pi/2)} \begin{pmatrix} 8 \\ 7 \end{pmatrix}$

Solution 28. We have

$$R_\theta(u) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus

$$\begin{aligned} R_{(3\pi/2)} \begin{pmatrix} 8 \\ 7 \end{pmatrix} &= \begin{pmatrix} \cos(3\pi/2) & -\sin(3\pi/2) \\ \sin(3\pi/2) & \cos(3\pi/2) \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ -8 \end{pmatrix}. \end{aligned}$$

Composition of Linear maps

Theorem 3.1.5. Let $T : \mathbb{R}^k \rightarrow \mathbb{R}^n$ and $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear maps such that T is induced by the matrix A and S is induced by the matrix B . Then $S \circ T$ is a linear map which is induced by the matrix BA .

$$\mathbb{R}^k \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Example 78. Let T be a linear map induced by the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ and S a linear map induced by the matrix $B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Find the matrix of the composite map $S \circ T$. Then, find $S \circ T \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Solution : the matrix of $S \circ T$ is given by BA .

$$BA = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 2 & 0 \end{pmatrix}$$

To find $S \circ T \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, multiply $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ by BA as follows

$$\begin{pmatrix} 8 & 4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 2 \end{pmatrix}$$

To check, first determine

$$T \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

Then, compute $S \left(T \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right)$ as follows :

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 24 \\ 2 \end{pmatrix}$$

3.1.7 The rank-nullity theorem

Definition 3.1.8. Let $A_{m \times n}$ be a matrix and $T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be a linear map. defined by $T(v) = Av$, $v \in \mathbb{F}^n$. The null space of a matrix A , also referred to as the kernel of A , is defined as follows.

$$\text{null}(A) = \{v \in \mathbb{F}^n \mid Av = 0_{\mathbb{F}^m}\}$$

Definition 3.1.9. The dimension of the null space of a matrix is called the **nullity**, denoted $\dim(\text{null}(A))$.

▷ The **rank of a matrix** A is the maximum number of linearly independent rows (or columns). Equivalently, it is the dimension of the column space (or row space).

▷ The rank of a matrix A , written $\text{rank}A$, is equal to the number of pivots in an echelon form of A . It is equal to the number of non-zero rows in its echelon form.

The rank is a very important property of a matrix and, depending on the context in which the matrix is used; it will be defined in many different ways. Of course, all the definitions lead to the same number.

▷ The **nullity** is the dimension of the **null space** (solutions to $Ax = 0$).

Theorem 3.1.6. Let $A_{m \times n}$ be a matrix over \mathbb{F} . Then the null space of A is a subspace of \mathbb{F}^n .

Theorem 3.1.7. Let $A_{m \times n}$ be a matrix. Then

$$\text{rank}(A) + \dim(\text{null}(A)) = n.$$

Proposition 3.1.5. Let $A_{m \times n}$ be a matrix over \mathbb{F} . Define $T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by $T(v) = Av$. Then

1. T is linear map
2. $\text{Im}(T) = \text{Col}(A)$
3. $\text{rank}(T) = \text{rank}(A)$
4. $\text{ker}(T) = \text{null}(A)$

Example 79. Find the rank of

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 3 & 5 & 4 & 3 \\ 1 & -1 & -1 & 2 & 1 \end{pmatrix}$$

Solution :

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 3 & 5 & 4 & 3 \\ 1 & -1 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{\substack{R_4 - R_1 \\ R_3 - 3R_1 \\ R_2 - 2R_1}} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & -3 & -4 & 1 & -3 \\ 0 & -3 & -4 & 1 & -3 \\ 0 & -3 & -4 & 1 & -3 \end{pmatrix} \xrightarrow{\substack{R_4 - R_2 \\ R_3 - R_2}} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & -3 & -4 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 1 & 4/3 & -1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is echelon form of matrix A and the number of its nonzero rows is 2, therefore, the rank of A is 2.

Example 80. Find the rank and nullity of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$.

Solution : Row reduce :

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{R_2 - 2R_1, R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus rank = 1. Since there are 3 columns, nullity = 3 - 1 = 2.

Exercise 29. In \mathbb{R}^4 , consider the vectors

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$$

Find a basis for the subspace $W = \text{span}\{u_1, u_2, u_3, u_4\}$. Determine its dimension.

Solution 29.

Form a matrix with these vectors as rows (or columns). We'll use rows for simplicity :

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 \\ 3 & 3 & 1 & 1 \end{pmatrix}$$

Row reduce :

$$\begin{aligned} R_2 &\leftarrow R_2 - 2R_1 : \\ R_3 &\leftarrow R_3 - R_1 : \\ R_4 &\leftarrow R_4 - 3R_1 : \end{aligned} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & -3 & -2 & 1 \end{pmatrix}$$

Subtract R_2 from R_3 and R_4 :

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus only the first two rows are non-zero. Hence a basis for W consists of u_1 and u_2 (since they are linearly independent). Alternatively, the reduced rows $(1, 2, 1, 0)$ and $(0, -3, -2, 1)$ are also a basis. The dimension is 2.

3.2 Change of Basis

The matrix of a linear map on a vector space \mathbb{V} depends both on the map itself and on the basis we choose for \mathbb{V} . Some bases make it easy to compute coordinates (components) of vectors, while others make the maps we are interested in look simple. Unfortunately, these bases are usually not the same, so we need to understand how to convert from one basis to another.

1- Operators

We begin with operators. (i.e : $T : \mathbb{V} \rightarrow \mathbb{V}$). Suppose \mathbb{V} is a vector space with bases \mathcal{B} and \mathcal{D} , that T is an operator on \mathbb{V} , and that we have computed the matrix $[T]_{\mathcal{B}}$ (matrix associated to T with respect to the basis \mathcal{B}). Now, for whatever reason, we wish to change our basis from \mathcal{B} to \mathcal{D} . How does that change our matrix?

The situation is described in **diagram below**.

$$\begin{array}{ccc}
 [v]_{\mathcal{B}} & \xrightarrow{[T]_{\mathcal{B}}} & [T(v)]_{\mathcal{B}} \\
 \downarrow P_{\mathcal{D}\mathcal{B}} & & \downarrow P_{\mathcal{D}\mathcal{B}} \\
 [v]_{\mathcal{D}} & \xrightarrow{[T]_{\mathcal{D}}} & [T(v)]_{\mathcal{D}}
 \end{array}$$

We are looking for a matrix $[T]_{\mathcal{D}}$ that converts the coordinates $[v]_{\mathcal{D}}$ of v in the \mathcal{D} basis into the coordinates $[T(v)]_{\mathcal{D}}$. We already know that multiplication by the change of basis matrix $P_{\mathcal{B}\mathcal{D}}$ converts $[v]_{\mathcal{D}}$ to $[v]_{\mathcal{B}}$, that multiplication by $[T]_{\mathcal{B}}$ converts $[v]_{\mathcal{B}}$ to $[T(v)]_{\mathcal{B}}$, and that multiplication by the matrix $P_{\mathcal{D}\mathcal{B}}$ converts $[T(v)]_{\mathcal{B}}$ to $[T(v)]_{\mathcal{D}}$. The product of the three matrices therefore converts $[v]_{\mathcal{D}}$ to $[T(v)]_{\mathcal{D}}$. This product therefore equals $[L]_{\mathcal{D}}$. In other words, we have proven :

Theorem 3.2.1. *Let T be an operator on a vector space with alternate bases \mathcal{B} and \mathcal{D} . The matrix of T relative to \mathcal{D} is given by*

$$[T]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}[T]_{\mathcal{B}}P_{\mathcal{B}\mathcal{D}}.$$

$P_{\mathcal{B}\mathcal{D}}$ define the change-of-basis matrix from \mathcal{B} to \mathcal{D} .

Since $[P]_{\mathcal{B}\mathcal{D}}^{-1} = [P]_{\mathcal{D}\mathcal{B}}$, we can also write this result as

$$[T]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}[T]_{\mathcal{B}}P_{\mathcal{D}\mathcal{B}}^{-1}$$

or as

$$[T]_{\mathcal{D}} = P_{\mathcal{B}\mathcal{D}}^{-1}[T]_{\mathcal{B}}P_{\mathcal{B}\mathcal{D}}$$

Example 81. *Let $\mathbb{V} = \mathbb{R}^2$ with the canonical basis $\mathcal{B} = \{e_1, e_2\}$ and let $\mathcal{D} = \{d_1 = e_1 + e_2, d_2 = e_1 - e_2\}$ be the alternate basis .*

The change of basis matrices are

$$P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_{\mathcal{D}\mathcal{B}} = P_{\mathcal{B}\mathcal{D}}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

If T is an operator that takes e_1 to $d_1 = 2e_1 + e_2$ and takes e_2 to $d_2 = e_1 + 2e_2$, then the matrix of T in the \mathcal{B} basis is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

while the matrix of T in the new basis \mathcal{D} is given by

$$\begin{aligned} [T]_{\mathcal{D}} &= P_{\mathcal{D}\mathcal{B}}[T]_{\mathcal{B}}P_{\mathcal{B}\mathcal{D}} \\ &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Remark 3.2.1. We can see that in the basis \mathcal{D} we have

$$T(d_1) = [T]_{\mathcal{D}}[d_1]_{\mathcal{D}} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3d_1$$

and

$$T(d_2) = [T]_{\mathcal{D}}[d_2]_{\mathcal{D}} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1d_2$$

In this example, the basis \mathcal{D} makes our operator look very simple. In fact, every operator can be made to look simple through the right choice of basis.

2- Linear map

Theorem 3.2.2. Let \mathbb{U} be a vector space with bases \mathcal{B} and \mathcal{D} , let \mathbb{V} be a vector space with bases \mathcal{B}' and \mathcal{D}' , and let $T : \mathbb{U} \rightarrow \mathbb{V}$ be a linear map. Let $[T]_{\mathcal{B}'\mathcal{B}}$ be the matrix of T relative to the \mathcal{B} and \mathcal{B}' bases. The matrix of T relative to the \mathcal{D} and \mathcal{D}' bases is given by

$$[T]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}}$$

Example 82.

Let $\mathbb{U} = \mathbb{R}^2$ with the canonical basis $\mathcal{B} = \{e_1, e_2\}$ and the alternate basis $\mathcal{D} = \{e_1 + e_2, e_1 - e_2\}$, and let $\mathbb{V} = \mathbb{R}^3$ with the canonical basis $\mathcal{B}' = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and the alternate basis $\mathcal{D}' = \{\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_3\}$.

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear map defined by :

$$\forall (x, y) \in \mathbb{R}^2, \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y - x \\ x \end{pmatrix}$$

Find the matrix relative to the \mathcal{D} and \mathcal{D}' , $([T]_{\mathcal{D}'\mathcal{D}})$.

Solution :

The matrix of T relative to \mathcal{B} and \mathcal{B}' is

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$$

The change of basis matrices are

$$P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_{\mathcal{D}\mathcal{B}} = P_{\mathcal{B}\mathcal{D}}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

and

$$P_{\mathcal{B}'\mathcal{D}'} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{\mathcal{D}'\mathcal{B}'} = P_{\mathcal{B}'\mathcal{D}'}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$$

while the matrix relative to the \mathcal{D} and \mathcal{D}' bases is given by

$$\begin{aligned} [T]_{\mathcal{D}'\mathcal{D}} &= P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Exercise 30 (see also ??). Consider the matrix :

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix}$$

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear map associated with matrix A with respect to the standard bases (canonical bases) of \mathbb{R}^3 and \mathbb{R}^2 , denoted by $\mathcal{B} = \{e_1, e_2, e_3\}$ and $\mathcal{B}' = \{f_1, f_2\}$ respectively. Let :

$$\begin{cases} e'_1 = e_2 + e_3, \\ e'_2 = e_1 + e_3, \\ e'_3 = e_1 + e_2, \end{cases} \quad \text{and} \quad \begin{cases} f'_1 = \frac{1}{2}(f_1 + f_2), \\ f'_2 = \frac{1}{2}(f_1 - f_2) \end{cases} .$$

1. Show that $\mathcal{B}' = \{e'_1, e'_2, e'_3\}$ is a basis of \mathbb{R}^3 and that $\mathcal{D}' = \{f'_1, f'_2\}$ is a basis of \mathbb{R}^2 .
2. Give the matrix associated with T in this new basis.

Solution 30.

1. Readers may verify for themselves that $\{e'_1, e'_2, e'_3\}$ and $\{f'_1, f'_2\}$ are bases of \mathbb{R}^3 and \mathbb{R}^2 , respectively

2. The transition matrix of \mathcal{B} to \mathcal{B}' is given by :

$$[P]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Also, the transition matrix from \mathcal{D} to \mathcal{D}' is given by :

$$[P]_{\mathcal{D}\mathcal{D}'} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We also know that matrix $[T]_{\mathcal{D}\mathcal{D}'}$ in the new basis is given by the formula

$$[T]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}}.$$

From different ways to find $[P]_{\mathcal{D}'\mathcal{B}'}$, here we choose to express the elements of the basis \mathcal{B}' in terms of the basis \mathcal{D}' , which allows us to determine $[P]_{\mathcal{D}'\mathcal{B}'}$. We have

$$\begin{cases} f'_1 = \frac{1}{2}(f_1 + f_2), \\ f'_2 = \frac{1}{2}(f_1 - f_2) \end{cases} \implies \begin{cases} f_1 = f'_1 + f'_2, \\ f_2 = f'_1 - f'_2 \end{cases}.$$

So

$$[P]_{\mathcal{D}'\mathcal{B}'} = [P]_{\mathcal{B}'\mathcal{D}'}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Thus,

$$[T]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$[T]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} -1 & 3 & 6 \\ 1 & 3 & -4 \end{pmatrix}.$$

3.3 Determinant

Let A be a square matrix. The determinant of A , denoted by $\det(A)$ is a very important number which we will explore throughout this section. If A is a 2×2 matrix, the determinant is given by the following formula.

Definition 3.3.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - cb$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

The 2×2 determinant can be used to find the determinant of larger matrices. We will now explore how to find the determinant of a 3×3 matrix, using several tools including the 2×2 determinant. We begin with the following definition.

The ij^{th} Minor of a Matrix

Definition 3.3.2. The determinant of a matrix A obtained by eliminating the i^{th} row and j^{th} column of A is called the ij^{th} Minor of A , denote M_{ij} .

Hence, there is a minor associated with each entry of A . Consider the following example which demonstrates this definition.

Example 83. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$. Find minor A_{12} and minor A_{23} .

Solution : First we will find minor A_{12} . By Definition, this is the determinant of the 2×2 matrix which results when you delete the first row and the second column. This minor is given by

$$M_{12} = \det \begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix} = (4)(1) - (3)(2) = -2.$$

Therefore $M_{12} = -2$.

Similarly, $minor(A)_{23}$ is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$M_{23} = \det \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = -4.$$

The ij^{th} Cofactor of a Matrix

Definition 3.3.3. Let A be an $n \times n$ matrix. The ij^{th} cofactor, denoted by $cof(A)_{ij}$ is defined to be

$$cof(A)_{ij} = (-1)^{i+j} M_{ij}$$

Example 84. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$. Find $cof(A)_{12}$ and $cof(A)_{23}$.

Solution :

$$cof(A)_{12} = (-1)^{1+2} M_{12} = -(-2) = 2$$

$$cof(A)_{23} = (-1)^{2+3} M_{23} = -(-4) = 4$$

Determinant of a Matrix

Definition 3.3.4. Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ matrix, the determinant of A , is given by

$$\det(A) = \sum_{j=1}^n a_{ij} cof(A)_{ij}$$

This process when applied to the i^{th} row (column) is known as expanding along the i^{th} row (column) as is given by

When calculating the determinant, you can choose to expand any row or any column. Regardless of your choice, you will always get the same number which is the determinant

of the matrix A . This method of evaluating a determinant by expanding along a row or a column is called **Laplace Expansion** or **Cofactor Expansion**.

Consider the following example.

Example 85. Find $\det(A) = \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

Solution :

1. First, we will calculate $\det(A)$ by expanding along the first column. Using Definition (3.3.4).

$$\begin{aligned} \det(A) &= 1 \overbrace{(-1)^{1+1}}^{cof(A)_{11}} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 4 \overbrace{(-1)^{2+1}}^{cof(A)_{21}} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 3 \overbrace{(-1)^{3+1}}^{cof(A)_{31}} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \\ &= -1 + 16 - 15 = 0 \end{aligned}$$

2. we can choose to expand along any row or column. Let's try now by expanding along the second row. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows.

$$\begin{aligned} \det(A) &= 4 \overbrace{(-1)^{2+1}}^{cof(A)_{21}} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 3 \overbrace{(-1)^{2+2}}^{cof(A)_{22}} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2 \overbrace{(-1)^{2+3}}^{cof(A)_{23}} \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ &= 16 - 24 - 8 = 0 \end{aligned}$$

You can see that for both methods, we obtained $\det(A) = 0$.

Example 86. Compute $\det A$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 3 & 1 \end{pmatrix}$.

Solution : Expand along first row :

$$\begin{aligned} \det A &= 1 \cdot \det \begin{pmatrix} -1 & 4 \\ 3 & 1 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \\ &= 1[(-1)(1) - 4(3)] - 2[0(1) - 4(2)] + 3[0(3) - (-1)(2)] \\ &= 1(-1 - 12) - 2(0 - 8) + 3(0 + 2) = -13 + 16 + 6 = 9. \end{aligned}$$

Example 87. Find $\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{pmatrix}$.

Solution :

As in the case of a 3×3 matrix, you can expand this along any row or column. Lets pick the third column. Then, using Laplace Expansion,

$$\det(A) = \overbrace{3(-1)^{1+3}}^{\text{cof}(A)_{13}} \begin{vmatrix} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + \overbrace{2(-1)^{2+3}}^{\text{cof}(A)_{23}} \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + \overbrace{4(-1)^{3+3}}^{\text{cof}(A)_{33}} \begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{vmatrix} + \overbrace{(-1)^{4+3}}^{\text{cof}(A)_{43}} \begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

Now, you can calculate each 3×3 determinant using Laplace Expansion, as we did above. You should complete these as an exercise and verify that $\det(A) = -12$.

The following provides a formal definition for the determinant of an $n \times n$ matrix. You may wish to take a moment and consider the above definitions for 2×2 and 3×3 determinants in context of this definition.

Definition 3.3.5. Let A be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$\det(A) = \sum_{j=1}^n a_{ij} \text{cof}(A)_{ij} = \sum_{i=1}^n a_{ij} \text{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

3.3.1 Properties

Properties 3.3.1.

1. **Triangular Matrices :**

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \prod_{i=1}^n a_{ii}$$

2. Let A be a matrix where tA is the transpose of A . Then,

$$\det({}^tA) = \det(A)$$

3. Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$. If this is true, it follows that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

4. If A is an $n \times n$ matrix such that one of its rows/columns consists of zeros, then

$$\det(A) = 0.$$

5. Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B)$$

6. If A is an $n \times n$ matrix and $\lambda \in \mathbb{R}$. Then

$$\det(\lambda A) = \lambda^n \det(A)$$

7. Swapping two rows changes the sign of the determinant.

8. Adding a multiple of one row to another does not change the determinant.

9. Multiplying a row by a scalar multiplies the determinant by that scalar.

Exercise 31. Find the inverse of $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 4 \end{pmatrix}$ using the adjugate formula.

Solution 31. The adjugate formula is given by

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

where $\text{adj}(A) = (\text{cof}(A)_{ij})^t$.

Step1 :

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 4 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ 2 & 0 & 1 & 2 \end{pmatrix} \quad C_4 \leftarrow C_4 - C_1. \\ &= \det \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} = 2 \end{aligned}$$

Step2 : Cofactors C_{ij}

$$\text{cof}(A)_{ij} = \begin{pmatrix} + \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{vmatrix} & - \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 4 \end{vmatrix} & + \begin{vmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 4 \end{vmatrix} & - \begin{vmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} \\ - \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{vmatrix} & + \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 4 \end{vmatrix} & + \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} \\ + \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} & + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 2 & 0 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\ - \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -4 & 2 & -16 & 6 \\ -1 & 1 & -2 & 1 \\ 2 & 0 & 4 & -2 \\ 1 & -1 & 4 & -1 \end{pmatrix}$$

Step3 :

$$\text{adj}(A) = (\text{cof}(A)_{ij})^t = \begin{pmatrix} -4 & -1 & 2 & 1 \\ 2 & 1 & 0 & -1 \\ -16 & -2 & 4 & 4 \\ 6 & 1 & -2 & -1 \end{pmatrix}$$

Therefore

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{2} \begin{pmatrix} -4 & -1 & 2 & 1 \\ 2 & 1 & 0 & -1 \\ -16 & -2 & 4 & 4 \\ 6 & 1 & -2 & -1 \end{pmatrix}$$

3.3.2 Gaussian elimination

Gaussian elimination is a useful and easy way to compute the inverse of a matrix. To compute a matrix inverse using this method, an augmented matrix is first created with the left side being the matrix to invert and the right side being the identity matrix. Then, Gaussian elimination is used to convert the left side into the identity matrix, which causes the right side to become the inverse of the input matrix.

Let $A \in \mathcal{M}_n$ be a square matrix. The method is given by augment with I_n and row reduce :

$$(A | I_n) \xrightarrow{\text{Elementaries operations}} (I_n | A^{-1})$$

Example 88. Find the inverse of

$$\begin{pmatrix} -1 & 3/2 \\ 1 & -1 \end{pmatrix}$$

Solution :

$$\begin{aligned} \left(\begin{array}{cc|cc} -1 & 3/2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array} \right) &\xrightarrow{R_1 + R_2 \rightarrow R_2} \left(\begin{array}{cc|cc} -1 & 3/2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right) \xrightarrow{R_1 - 3R_2 \rightarrow R_1} \left(\begin{array}{cc|cc} -1 & 0 & -2 & -3 \\ 0 & 1/2 & 1 & 1 \end{array} \right) \\ &\xrightarrow{\substack{-R_1 \rightarrow R_1 \\ 2R_2 \rightarrow R_2}} \left(\begin{array}{cc|cc} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 2 \end{array} \right) \end{aligned}$$

Thus

$$A^{-1} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$$

Example 89. Find the inverse of

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Solution :

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_3 - 2R_1 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & -2 & 1 \end{array} \right)$$

$$\begin{aligned} & \xrightarrow{R_3 + R_2 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array} \right) \xrightarrow{R_3 + R_2 \rightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array} \right) \\ & \xrightarrow{R_1 - R_2 + R_3 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & -1 & 1 & -2 & - \end{array} \right) \xrightarrow{-R_3 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -1 & 2 & 2 \end{array} \right) \end{aligned}$$

Thus

$$A^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -2 & 1 \\ -1 & 2 & -2 \end{pmatrix}$$

Linear Systems

4.1 Introduction to Linear Systems and Matrices

We now introduce linear equations, linear systems, and matrices.

4.1.1 Linear equations and linear systems

We consider

$$a_1x + a_2x^2 + \cdots + a_nx^n = b,$$

where $a_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$) are coefficients, x_i ($i = 1, 2, \dots, n$) are variables (unknowns), n is a positive integer, and $b \in \mathbb{R}$ is a constant.

Definition 4.1.1. A **linear system** (or system of linear equations) is a collection of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

When $b_i = 0$, $\forall 1 \leq i \leq n$, the linear system is called a **homogeneous linear system**.

The goal is to find all n -tuples (x_1, \dots, x_n) that satisfy all equations simultaneously. The set of all solutions of the equation is called the **solution set of the equation**.

Example 90. Consider the system

$$\begin{cases} 2x - 3y = 5 \\ -x + 4y = 2 \end{cases}$$

This is a linear system with two equations and two unknowns. It can be solved by substitution or elimination. For instance, multiplying the second equation by 2 and adding to the first yields $5y = 9$, so $y = 9/5$, and then $x = 2 + 4y = 2 + 36/5 = 46/5$, thus the solution set

is

$$\mathbf{S} = \{(46/5, 9/5)\}$$

Example 91. Consider the following linear system

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2. \end{cases}$$

The graphs of these equations are lines called l_1 and l_2 . We have three possible cases of lines l_1 and l_2 in xy -plane. See Figure 1.1.

- When l_1 and l_2 are parallel, there is no solution of the system.
- When l_1 and l_2 intersect at only one point, there is exactly one solution of the system.
- When l_1 and l_2 coincide, there are infinitely many solutions of the system.

Example 92. We consider the following linear equations :

$$\begin{aligned} (a) \quad & x + y = 1. \\ (b) \quad & x + y + z = 1. \end{aligned}$$

It is easy to see that the solution set of (a) is a line in xy -plane and the solution set of (b) is a plane in xyz -space.

Definition 4.1.2. Two linear systems with m equations in n unknowns are equivalent to each other if they have the same solutions.

4.1.2 Matrix Representation

The system can be written compactly using matrices. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then the system is

$$A\mathbf{x} = \mathbf{b}.$$

Here A is called the **coefficient matrix**, \mathbf{x} the **unknown vector**, and \mathbf{b} the **constant vector**. If $m = n$ (square matrix), we often look for a unique solution.

Corollary 4.1.1. For a linear system $A\mathbf{x} = \mathbf{b}$:

- If $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$, the system is consistent.
- If $\text{rank}(A) = n$ (full column rank), the solution is unique.
- If $\text{rank}(A) < n$, there are infinitely many solutions (if consistent), parameterized by $n - \text{rank}(A)$ free variables.

Example 93. The system

$$\begin{cases} 2x - 3y = 5 \\ -x + 4y = 2 \end{cases}$$

becomes

$$\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

So

$$\begin{cases} 2x - 3y = 5 \\ -x + 4y = 2 \end{cases} \iff \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

4.2 Solving Linear Systems

There are several methods to solve linear systems. We present the most important ones.

4.2.1 Gaussian Elimination and Gauss-Jordan Elimination

a- Augmented matrix

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then the system is

$$A\mathbf{x} = \mathbf{b}.$$

Definition 4.2.1. The matrix which is made up of the coefficient matrix A and the right-hand side \mathbf{b} is called the augmented matrix of the system and is denoted by $[A|\mathbf{b}]$. For a linear system of m equations in n unknowns the augmented matrix is the $m \times (n + 1)$ matrix :

$$[A|\mathbf{b}] = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

Example 94. Derive the augmented matrix for the following linear system.

$$\begin{cases} 2x + 3y = 7 \\ x - 4y - z = -2 \\ 2y - 3z = 1 \end{cases}$$

Solution : We have the coefficient, unknown, and right-hand side matrices :

$$\begin{cases} 2x + 3y = 7 \\ x - 4y - z = -2 \\ 2y - 3z = 1 \end{cases} \iff \underbrace{\begin{pmatrix} 2 & 3 & 0 \\ 1 & -4 & -1 \\ 0 & 2 & -3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_x = \underbrace{\begin{pmatrix} 7 \\ -2 \\ 1 \end{pmatrix}}_b$$

so the augmented matrix is :

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 3 & 0 & 7 \\ 1 & -4 & -1 & -2 \\ 0 & 2 & -3 & 1 \end{array} \right).$$

b- Row Echelon Form

Definition 4.2.2. The leftmost, nonzero entry in a row of a matrix is called the leading entry or pivot.

Example 95. Consider the linear system

$$\begin{cases} 2x + 3y = 7 \\ 4y - z = -2 \\ -3z = 1 \end{cases}.$$

Solution :

So its corresponding augmented matrix is therefore

$$(A|b) = \left(\begin{array}{ccc|c} \boxed{2} & 3 & 0 & 7 \\ 0 & \boxed{4} & -1 & -2 \\ 0 & 0 & -3 & 1 \end{array} \right).$$

$\boxed{2}$ and $\boxed{4}$ are two pivots .

Definition 4.2.3. A matrix is in row echelon form (**REF**) if :

1. All zero rows (i.e. rows consisting entirely of zeros) are at the bottom of the matrix.
2. All elements below a leading entry (pivot) are zero.
3. Each leading entry is to the right of the leading entries of all rows above it.
4. Each leading entry is equal to 1. Such an entry is called the leading 1.

Examples 1. These matrices are in (**REF**) form,

$$\left(\begin{array}{ccc} 1 & 2 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array} \right), \quad \left(\begin{array}{cccc} 1 & 2 & -3 & 3 \\ 0 & 1 & -5 & -2 \end{array} \right)$$

c- Reduced Row Echelon Form

Definition 4.2.4. A matrix is in reduced row echelon form (**RREF**) if it is in row echelon form and the leading one is the only nonzero entry in its column.

Examples 2. These matrices are in **RREF** form,

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array} \right), \quad \left(\begin{array}{ccc} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array} \right)$$

Exercise 32. Determine if the matrices are in **REF**, **RREF**, or neither.

$$\begin{pmatrix} 1 & 8 & -3 \\ 0 & \sqrt{2} & \pi \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \pi & 2 & 3/2 \\ 0 & 1 & -2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution 32.

$$\begin{pmatrix} 1 & 8 & -3 \\ 0 & \sqrt{2} & \pi \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{REF}, \quad \begin{pmatrix} 1 & \pi & 2 & 3/2 \\ 0 & 1 & -2 & 4 \end{pmatrix} \Rightarrow \mathbf{REF}, \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{pmatrix} \Rightarrow \mathbf{RREF},$$

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{RREF}, \quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \mathbf{Neither form}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{Neither form}$$

d- Gaussian Elimination (Row Reduction)

• Elementary Row Operations

Performing one of the following three operations, called **elementary row operations** on a given linear system gives an equivalent linear system.

1. Swap two rows or Interchange two equations (or rows) $R_i \leftrightarrow R_j$.
2. Multiply a row by a non-zero scalar $R_i \rightarrow cR_j$,
3. Add a multiple of one row to another $R_i \rightarrow R_i + cR_j$.

Here R_i denotes a row of an augmented matrix and c represents a scalar.

Note : Every augmented matrix can be reduced to its row echelon form (**REF**) using elementary row operations. This process is called **Gaussian elimination**. Every augmented matrix can be reduced to its reduced row echelon form (**RREF**) by a process called **Gauss-Jordan elimination**.

• Steps in Gaussian Elimination

1. Locate the leftmost nonzero column in the augmented matrix. If the top entry of the column is zero, interchange the top row with another so the top entry (call it a) is nonzero.
2. If a is not a leading 1 make it so by multiplying the row by $1/a$.
3. Make all other entries in the column below the leading 1 equal zero by adding suitable multiples of the first row to the remaining rows.
4. Consider the remaining matrix produced by ignoring the top row. Repeat steps 1 – 4 on that matrix. If the remaining matrix has no rows the original matrix is now in row echelon form.

Note that once the matrix is in row echelon form the system may be solved as shown in the following examples.

Example 96. Solve

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases}$$

Solution : The augmented matrix for the system is

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array} \right)$$

Since the top left entry is already a leading 1, zero the 2 below it by adding -2 times R_1 to R_2 . (This is the same as subtracting 2 times R_1 from R_2 .)

$$R_2 \leftarrow R_2 + (-2)R_1 \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array} \right)$$

Next zero the first entry in the third row by adding -3 times R_1 to R_3 .

$$R_3 \leftarrow R_3 - 3R_1 \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right)$$

To get a leading 1 in the second row (R_2), multiply the row by $1/2$, or equivalently, divide the row by 2.

$$R_2 \leftarrow \frac{1}{2}R_2 = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array} \right)$$

Zero the second entry in the third row by adding -3 times R_2 to R_3 .

$$R_3 \leftarrow R_3 - 3R_2 = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{array} \right)$$

Get a leading 1 in the third row by dividing the row by $-1/2$, or, equivalently, multiplying it by -2 .

$$R_3 \leftarrow -2R_3 = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

The linear system represented by the last augmented matrix is

$$\begin{cases} x + y + 2z = 9 \\ y - \frac{7}{2}z = -\frac{17}{2} \\ z = 3 \end{cases}$$

so back-substitution gives the following

$$x = 1, \quad y = 2 \quad \text{and} \quad z = 3$$

$$\mathbf{S}\{(x = 1, y = 2, z = 3)\}$$

which is easily checked in the original system.

Example 97. Solve the linear system

$$\begin{cases} x + 2y + z = 4 \\ 2x + y - z = 1 \\ 3x - y + 2z = 3 \end{cases}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & 2 & 3 \end{array} \right).$$

Row operations :

$$\begin{aligned} R_2 &\leftarrow R_2 - 2R_1 : & (0 & -3 & -3 & -7) \\ R_3 &\leftarrow R_3 - 3R_1 : & (0 & -7 & -1 & -9) \end{aligned}$$

Now we have

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & -3 & -7 \\ 0 & -7 & -1 & -9 \end{array} \right).$$

Row operations :

$$\begin{aligned} R_2 &\leftarrow -\frac{1}{3}R_2 : & (0 & 1 & 1 & \frac{7}{3}) \\ R_3 &\leftarrow R_3 + 7R_2 : & (0 & 0 & 6 & \frac{22}{3}) \end{aligned}$$

we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 7/3 \\ 0 & 0 & 6 & 22/3 \end{array} \right).$$

Now

- $6z = 22/3 \Rightarrow z = 11/9$.
- From $y + z = 7/3$, we get $y = 10/9$.
- Finally $x + 2y + z = 4 \Rightarrow x = 5/9$.

The set Solution :

$$\mathbf{S} \{(x = 5/9, y = 10/9, z = 11/9)\}.$$

Exercise 33. Use Gaussian elimination to solve the system

$$\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 7 \\ 3x + 2y - z = 4 \end{cases}$$

Solution 33. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 7 \\ 3 & 2 & -1 & 4 \end{array} \right).$$

Row operations :

$$\begin{aligned} R_2 &\leftarrow R_2 - 2R_1 \\ R_3 &\leftarrow R_3 - 3R_1 \end{aligned} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & -5 \\ 0 & -1 & -4 & -14 \end{array} \right).$$

$$\begin{aligned} R_2 &\leftarrow -\frac{1}{3}R_2 \\ R_3 &\leftarrow R_3 + \frac{1}{3}R_2 \end{aligned} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & -13/3 & -37/3 \end{array} \right).$$

we get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & -13/3 & -37/3 \end{array} \right).$$

Now

$$-\frac{13}{3}z = -\frac{37}{3} \Rightarrow z = \frac{37}{13}.$$

$$\text{— From } y - \frac{1}{3}z = \frac{5}{3}, \text{ we get } y = \left(\frac{1}{3}\right)\left(\frac{37}{13}\right) + \frac{5}{3} = \frac{34}{13}.$$

$$\text{— Finally } x + y + z = 6 \Rightarrow x = 6 - \frac{37}{13} - \frac{34}{13} = \frac{7}{13}.$$

The set Solution :

$$\mathbf{S}\{(x = 7/13, y = 34/13, z = 37/13)\}.$$

Exercise 34. Solve in \mathbb{R}^4 the following linear system

$$\begin{cases} w - x + 2y - z = -1 \\ 2w + x - 2y - 2z = -2 \\ -w + 2x - 4y + z = 1 \\ 3w - 3z = -3 \end{cases}$$

Solution 34. we have

$$(A|b) = \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array} \right)$$

$$\begin{aligned} R_2 &\leftarrow R_2 - 2R_1 \\ R_3 &\leftarrow R_3 + R_1 \\ R_4 &\leftarrow R_4 - 3R_1 \end{aligned} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array} \right)$$

and

$$\begin{aligned} R_2 &\leftarrow \frac{1}{3}R_2 \\ R_3 &\leftarrow R_3 - R_2 \\ R_4 &\leftarrow R_4 - 3R_2 \end{aligned} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array} \right) \begin{aligned} R_3 &\leftarrow R_3 - R_2 \\ R_4 &\leftarrow R_4 - 3R_2 \end{aligned} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The linear system corresponding to the row echelon form is then

$$\begin{cases} w - x + 2y - z = -1 \\ x - 2y = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

This system will have an infinite number of solutions. To characterize them we will introduce parameters.

Definition 4.2.5. The variables that correspond to the leading entries of the row echelon form of an augmented matrix are called the leading variables or dependent variables. The remaining variables are called the **free variables** or **independent variables**

To solve a system of linear equations, set the free variables equal to parameters and use the row echelon form with back-substitution to solve for the leading variables.

Complete the solution to the **exercise (34)**.

The row echelon form found and corresponding linear system are

$$\left(\begin{array}{cccc|c} \boxed{1} & -1 & 2 & -1 & 1 \\ 0 & \boxed{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \iff \begin{cases} w - x + 2y - z = -1 \\ x - 2y = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

The leading entries are boxed and are found in the w and x variable columns. Thus w and x are the dependent variables and y and z are the independent variables. So introduce two parameters for the independent variables letting $y = \alpha$ and $z = \beta$. Next solve for the dependent variables in terms of the parameters using back-substitution.

$$x = 2\alpha, \quad w = \beta - 1$$

We can write the solution to the system as

$$w = \beta - 1, \quad x = 2\alpha, \quad y = \alpha, \quad z = \beta,$$

where α and β are parameters taking on any numbers.

We can also write the solution in the form :

$$S = \left\{ \left(\begin{array}{c} \beta - 1 \\ 2\alpha \\ \alpha \\ \beta \end{array} \right) \mid \alpha, \beta \in \mathbb{R} \right\}$$

e- Gauss-Jordan Elimination (Reduced Row Reduction)

Gauss-Jordan elimination takes Gaussian elimination one further step to produce an augmented matrix in reduced row echelon form. One does the following steps :

1. Perform Gaussian elimination to put the augmented matrix in row echelon form (**REF**).
2. Add suitable multiples of the last nonzero row to the rows above it to introduce zeros into them above the leading 1 of this row.
3. Consider the remaining matrix produced by ignoring the last nonzero row and any zero rows beneath it, if any. Repeat steps 2 – 3 on that matrix. If the remaining matrix has no rows the original matrix is now in reduced row echelon form (**RREF**).

Example 98. Solve the linear system using Gauss-Jordan elimination :

$$\begin{cases} -y + 5z = 9 \\ x + y + 2z = 8 \\ 3x - 7y + 4z = 10 \end{cases}$$

Solution : We have

$$(A|b) = \left(\begin{array}{ccc|c} 0 & -1 & 5 & 9 \\ 1 & 1 & 2 & 8 \\ 3 & -7 & 4 & 10 \end{array} \right)$$

First put the augmented matrix in row echelon form :

$$\begin{aligned} R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array} \right) &\Rightarrow R_3 \leftarrow R_3 - 3R_1 \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right) \\ R_3 \leftarrow R_3 - 10R_2 \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array} \right) &\Rightarrow \begin{array}{l} R_3 \leftarrow -\frac{1}{52}R_3 \\ R_2 \leftarrow -R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right) \quad (\mathbf{REF}) \end{aligned}$$

To achieve reduced row echelon form work from the bottom of the matrix upward, to get zeros above any leading one.

$$\begin{array}{l} R_3 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 + 5R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right) \Rightarrow R_1 \leftarrow R_1 - R_2 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right) \quad (\mathbf{RREF})$$

The unique solution to the system is therefore

$$x = 3, \quad y = 1, \quad z = 2$$

4.2.2 Inverse Matrix Method

If A is square and invertible, the solution is $\mathbf{x} = A^{-1}\mathbf{b}$. Computing the inverse can be done via Gaussian elimination on $[A|I]$.

Example 99. For the system $\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$,

We compute the inverse of $A = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$. The determinant

$$\det A = 2 \cdot 4 - (-3)(-1) = 8 - 3 = 5.$$

Then

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

check :

$$AA^{-1} = \frac{1}{5} \begin{pmatrix} 2 \cdot 4 + (-3) \cdot 1 & 2 \cdot 3 + (-3) \cdot 2 \\ (-1) \cdot 4 + 4 \cdot 1 & (-1) \cdot 3 + 4 \cdot 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = I.$$

Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 20 + 6 \\ 5 + 4 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 26 \\ 9 \end{pmatrix} = \begin{pmatrix} 26/5 \\ 9/5 \end{pmatrix}.$$

Example 100. Solve the system using the inverse matrix :

$$\begin{cases} 2x + 3y = 7 \\ x - 4y = -2 \end{cases}$$

Solution : Write as $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}.$$

Compute $\det A = 2(-4) - 3(1) = -8 - 3 = -11 \neq 0$, so A^{-1} exists :

$$A^{-1} = \frac{1}{-11} \begin{pmatrix} -4 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{2}{11} \end{pmatrix}.$$

Then

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{2}{11} \end{pmatrix} \begin{pmatrix} 7 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{28}{11} - \frac{6}{11} \\ \frac{7}{11} + \frac{4}{11} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

4.2.3 Cramer's Rule

For a square system $A\mathbf{x} = \mathbf{b}$ with $\det A \neq 0$, each variable x_j is given by

$$x_j = \frac{\det(A_j)}{\det(A)},$$

where A_j is the matrix obtained by replacing the j -th column of A by the column vector \mathbf{b} .

Example 101. Using the same matrix A and vector \mathbf{b} as above.

$$\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

We have

$$\det A = 5$$

$$A_1 = \begin{pmatrix} 5 & -3 \\ 2 & 4 \end{pmatrix}, \quad \det A_1 = 5 \cdot 4 - (-3) \cdot 2 = 20 + 6 = 26,$$

$$A_2 = \begin{pmatrix} 2 & 5 \\ -1 & 2 \end{pmatrix}, \quad \det A_2 = 2 \cdot 2 - 5 \cdot (-1) = 4 + 5 = 9.$$

Thus

$$x = \frac{\det A_1}{\det A} = 26/5, \quad y = \frac{\det A_2}{\det A} = 9/5.$$

Example 102. Consider

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

Solution :

- The rows are linearly dependent (second row is twice the first), so rank is 1.
- The nullity is 1. For $\mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$, the system is consistent because \mathbf{b} is also twice the first column combination.
- The general solution is $x + 2y = 3$, so $x = 3 - 2y$, with y free.

4.3 Eigenvalues and Eigenvectors

Definition 4.3.1. Given a square matrix A , a scalar λ and a non-zero vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

are called an **eigenvalue** and corresponding **eigenvector**. They satisfy the characteristic equation

$$\det(A - \lambda I) = 0.$$

Eigenvalues are important in many applications : diagonalization, stability analysis, differential equations, etc.

Example 103. Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Solution :

— We have

$$\det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

So eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$.

— For $\lambda = 1$: solve $(A - I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \Rightarrow v_1 + v_2 = 0, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ (up to scale).}$$

— For $\lambda = 3$: $(A - 3I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \Rightarrow -v_1 + v_2 = 0, \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Exercise 35.

1. Compute the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

2. Find the rank and nullity of $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$.

Solution 35.

1. Inverse of A : Augment with I_3 and row reduce :

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -5 & -3 \\ 0 & 1 & 0 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array} \right)$$

Hence

$$A^{-1} = \begin{pmatrix} 4 & -5 & -3 \\ -2 & 3 & 2 \\ 1 & -1 & -1 \end{pmatrix}.$$

2. $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ has rows that are multiples : $R_2 = 2R_1$, $R_3 = 3R_1$. Hence

$$\text{rank}(B) = 1, \quad \text{nullity} = 3 - 1 = 2.$$

Chapitre 5

Exercises

Exercise 36. Determine whether the set

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x - 2y + 3z = 0 \right\}$$

is a subspace of \mathbb{R}^3 .

Solution 36. We check the three subspace conditions :

1. **Zero vector :** $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies $0 - 2 \cdot 0 + 3 \cdot 0 = 0$, so, $\mathbf{0} \in S$.

2. **Closed under addition :** Let $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in S$.

We have

$$u \in S \Leftrightarrow x_1 - 2y_1 + 3z_1 = 0$$

$$v \in S \Leftrightarrow x_2 - 2y_2 + 3z_2 = 0$$

$$\text{Then } u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

$$(x_1 + x_2) - 2(y_1 + y_2) + 3(z_1 + z_2) = (x_1 - 2y_1 + 3z_1) + (x_2 - 2y_2 + 3z_2) = 0 + 0 = 0,$$

so, $u + v \in S$.

3. **Closed under scalar multiplication :** For any scalar $\alpha \in \mathbb{R}$. Then $\alpha u = \begin{pmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \end{pmatrix}$

$$(\alpha x_1) - 2(\alpha y_1) + 3(\alpha z_1) = \alpha(x_1 - 2y_1 + 3z_1) = \alpha \cdot 0 = 0,$$

so $\alpha u \in S$.

All conditions hold, so S is a subspace.

Exercise 37. Let $V = \mathbb{R}^3$ and let $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\}$. Find a basis for W and determine its dimension.

Solution 37.

The two vectors are linearly independent because neither is a scalar multiple of the other
check : if

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

then first component gives $1 = \lambda \cdot 0$ impossible). Thus they form a basis of W . Hence $\dim W = 2$ and a basis is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\}$.

Exercise 38. Determine whether the following subsets of \mathbb{R}^3 are subspaces. Justify your answers.

$$W_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}, \quad W_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \right\},$$

$$W_3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = 2y \text{ and } z = -y \right\}, \quad W_4 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : xy = 0 \right\}.$$

Solution 38.

1. W_1 is a subspace. Check :

— $0_{\mathbb{R}^3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies : $0 + 0 + 0 = 0$.

— If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in W_1$, then $x + y + z = 0$ and $x' + y' + z' = 0$. Adding gives

$$(x + x') + (y + y') + (z + z') = 0,$$

so

$$\begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix} \in W_1.$$

— If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W_1$ and $\alpha \in \mathbb{R}$, then

$$\alpha x + \alpha y + \alpha z = \alpha(x + y + z) = 0,$$

so

$$\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} \in W_1.$$

Hence W_1 is a subspace (in fact, a plane through the origin).

2. W_2 is not a subspace.

The zero vector $0_{\mathbb{R}^3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ does not satisfy : $0^2 + 0^2 + 0^2 = 1 \neq 1$. So it fails the first condition.

3. W_3 is a subspace. Conditions :

$$x = 2y \quad \text{and} \quad z = -y.$$

Then any vector in W_3 is of the form

$$\begin{pmatrix} 2y \\ y \\ -y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Check :

— $0_{\mathbb{R}^3}$ corresponds to $y = 0$.

— If y and y' give vectors $u = \begin{pmatrix} 2y \\ y \\ -y \end{pmatrix}$, $v = \begin{pmatrix} 2y' \\ y' \\ -y' \end{pmatrix}$, then

$$u + v = \begin{pmatrix} 2(y + y') \\ (y + y') \\ -(y + y') \end{pmatrix}$$

which is again of the same form.

— Scalar multiple $\alpha u = \begin{pmatrix} 2\alpha y \\ \alpha y \\ -\alpha y \end{pmatrix}$ also of the form.

Thus W_3 is a subspace (a line through the origin).

4. W_4 is not a subspace. Take $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Both satisfy $xy = 0$ because $1 \cdot 0 = 0$

and $0 \cdot 1 = 0$. But $u + v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ has $1 \cdot 1 = 1 \neq 0$, so not closed under addition. Hence not a subspace.

Exercise 39. Let V be the vector space of all real-valued functions defined on \mathbb{R} . Determine whether the following subsets are subspaces.

1. $U_1 = \{f \in V : f(0) = 0\}$.
2. $U_2 = \{f \in V : f(0) = 1\}$.
3. $U_3 = \{f \in V : f \text{ is continuous}\}$.

4. $U_4 = \{f \in V : f(x) \geq 0 \text{ for all } x\}$.

Solution 39.

1. U_1 is a subspace :

- Zero function satisfies $0(0) = 0$.
- If $f(0) = 0$ and $g(0) = 0$, then $(f + g)(0) = 0 + 0 = 0$.
- If $f(0) = 0$ and $\alpha \in \mathbb{R}$, then $(\alpha f)(0) = \alpha \cdot 0 = 0$.

So U_1 is a subspace.

2. U_2 is not a subspace because the zero function does not satisfy $0(0) = 1$.

3. U_3 is a subspace : the sum of continuous functions is continuous, and scalar multiples are continuous. The zero function is continuous. So yes.

4. U_4 is not a subspace : the condition $f(x) \geq 0$ is not closed under scalar multiplication with negative scalars. For example, $f(x) = 1$ (nonnegative) but $-1 \cdot f = -1$ is not nonnegative. Also closure under addition holds (sum of nonnegative functions is nonnegative), but scalar multiplication fails.

Exercise 40. Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, w_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$$

be vectors in \mathbb{R}^3

(a) Let $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$. Find a basis of $V \cap W$.

(b) Complete $\{w_1, w_2\}$ to a basis of \mathbb{R}^3 .

Solution 40.

(a) We have $V = \text{span}\{v_1, v_2\}$ The two vectors v_1, v_2 are linearly dependent because $v_2 = 2v_1$ so,

$$V = \text{span}\{v_1\} \iff V = \{\lambda v_1, \lambda \in \mathbb{R}\}$$

$W = \text{span}\{w_1, w_2\}$. The two vectors w_1, w_2 are linearly independent because neither is a scalar multiple of the other. so

$$W = \text{span}\{w_1, w_2\} \iff W = \{\alpha w_1 + \beta w_2, \alpha, \beta \in \mathbb{R}\}$$

Let $u \in \mathbb{R}^3$, we have

$$u \in V \cap W \iff u \in V \text{ and } u \in W$$

$$\iff \begin{cases} u = \lambda v_1 \\ u = \alpha w_1 + \beta w_2 \end{cases}$$

$$\iff \lambda v_1 = \alpha w_1 + \beta w_2,$$

$$\iff v_1, w_1, w_2 \text{ are linearly dependent or linearly independent}$$

— If $v_1 \in W$, v_1, w_1, w_2 are linearly dependent and $V \cap W = V$

— If $v_1 \notin W$, v_1, w_1, w_2 are linearly independent and $V \cap W = \{0_{\mathbb{R}^3}\}$

$$\alpha w_1 + \beta w_2 - \lambda v_1 = \{0_{\mathbb{R}^3}\} \Leftrightarrow \alpha \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For example if we take $\alpha = \beta = \gamma = 1$ we get

$$w_1 + w_2 - v_1 = \{0_{\mathbb{R}^3}\}$$

Therefore $v_1 \in W$ and v_1, w_1, w_2 are linearly dependent and $V \cap W = V$, then $\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

is a basis of $V \cap W$

(b) $\{w_1, w_2\} = \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \right\}$. They are independent. Add any vector not in their span, e.g., $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then $\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of \mathbb{R}^3 .

Exercise 41. Find the coordinates of the vector $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with respect to the basis $\{u_1, u_2\}$ of \mathbb{R}^2 , where

$$u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solution 41. let x, y the coordinates of v with respect to basis $\{u_1, u_2\}$, we have

$$\begin{aligned} v = xu_1 + yu_2 &\Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 3 \\ 2 \end{pmatrix} + y \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} 2 = 3x + 2y \\ 1 = 2x + 3y \end{cases} \Leftrightarrow v = \begin{pmatrix} 4/5 \\ -1/5 \end{pmatrix} \end{aligned}$$

Exercise 42. Let $V = \text{span}(v_1, v_2, v_3)$ (over $\mathbb{F} = \mathbb{R}$), where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

(-) Compute $\dim V$.

Solution 42.

(-) We have

$$V = \text{span} \{v_1, v_2, v_3\}$$

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0_{\mathbb{R}^3} \Leftrightarrow \begin{cases} \alpha + 2\beta + \gamma = 0 \\ \alpha + 2\beta + \gamma = 0 \\ \alpha + 2\beta + 2\gamma = 0 \\ \alpha + 3\beta - \gamma = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha + 2\beta + \gamma = 0 \\ \alpha + 2\beta + 2\gamma = 0 \\ \alpha + 3\beta - \gamma = 0 \end{cases} \Leftrightarrow \alpha = \beta = \gamma = 0$$

Since $\{v_1, v_2, v_3\}$ are linearly independent. Thus, $\{v_1, v_2, v_3\}$ is a basis of V , whereby $\dim V = 3$.

Exercise 43. In \mathbb{R}^3 , let

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

1. Determine whether $w = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ lies in the span of $\{v_1, v_2, v_3\}$.
2. Find a basis for the span of $\{v_1, v_2, v_3\}$.
3. What is the dimension of this span?

Solution 43.

1. We need scalars a, b, c such that

$$a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}.$$

This gives the system :

$$\begin{cases} a + 2b = 5 \\ b + c = 3 \\ a + c = 2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \\ c = 1 \end{cases}. \text{ So } w = 1v_1 + 2v_2 + 1v_3 \text{ Thus } w \text{ is in the span.}$$

2. To find a basis, we check linear independence. Form a matrix with these vectors as rows (or columns) and row reduce. Write them as rows :

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Row reduce :

$$(R_2 \leftarrow R_2 - 2R_1) \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix} \implies (R_3 \leftarrow R_3 - R_2) \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$$

The rows are linearly independent (no zero rows). Hence the three vectors are linearly independent and form a basis of their span.

3. Since we have three linearly independent vectors in \mathbb{R}^3 , they span all of \mathbb{R}^3 ? Wait, \mathbb{R}^3 has dimension 3, so three linearly independent vectors indeed form a basis of \mathbb{R}^3 . So the span is \mathbb{R}^3 itself, dimension 3.

Exercise 44. Determine whether the following sets of vectors in \mathbb{R}^3 are linearly independent.

1. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}.$

2. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right\}.$

3. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$

Solution 44.

1. Form a matrix with these as rows (or columns) and compute determinant if square. Here we have three vectors in \mathbb{R}^3 , so we can form a 3×3 matrix and check if determinant is zero. Matrix with vectors as columns :

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Compute determinant :

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow R_3 \leftarrow R_3 - R_1 : \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \rightarrow R_3 \leftarrow R_3 + 2R_2 : \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\det M = 0.$$

So they are linearly dependent. Indeed, the third is the sum of the first two ?

Check :

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

exactly. So dependent.

2. Two vectors are linearly dependent if one is a scalar multiple of the other. Here

$$\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

so they are linearly dependent.

3. Form matrix with these as columns :

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies R_2 \leftarrow R_2 - R_3 : \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Determinant :

$$\det M = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0.$$

Hence linearly independent.

Exercise 45. In $\mathbb{R}_2[t]$ (polynomials of degree ≤ 2), consider the polynomials

$$p_1(t) = 1 + t, \quad p_2(t) = t + t^2, \quad p_3(t) = 1 + t^2.$$

Are they linearly independent ?

Solution 45.

We need to check if

$$\alpha p_1 + \beta p_2 + \gamma p_3 = 0 \text{ (the zero polynomial)} \Rightarrow \alpha = \beta = \gamma = 0.$$

Write :

$$\alpha(1 + t) + \beta(t + t^2) + \gamma(1 + t^2) = (\alpha + \gamma) + (\alpha + \beta)t + (\beta + \gamma)t^2 = 0.$$

Thus we have the system :

$$\begin{cases} \alpha + \gamma = 0 \\ \alpha + \beta = 0 \\ \beta + \gamma = 0 \end{cases} \Rightarrow \alpha = \beta = \gamma = 0.$$

So they are linearly independent.

Exercise 46. Find a basis for the subspace $W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0\}$. What is its dimension ?

Solution 46.

The equation defines a plane through the origin. We can express $z = x + 2y$. Then any vector in W is of the form

$$\begin{pmatrix} x \\ y \\ x + 2y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Thus the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ span W . They are linearly independent because neither is a multiple of the other. Hence they form a basis, and $\dim W = 2$.

Exercise 47. Let V be the vector space of all 2×2 symmetric matrices (i.e., $A^T = A$). Find a basis for V and determine its dimension.

Solution 47.

A symmetric 2×2 matrix has the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. This can be written as

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

These three matrices are linearly independent : if a linear combination gives the zero matrix, then each coefficient must be zero. Thus a basis is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \text{ So Dimension is } 3.$$

Exercise 48. In \mathbb{R}^4 , find the dimension of the subspace spanned by

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Solution 48.

Form a matrix with these vectors as rows (or columns). We'll use rows :

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & -3 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Row reduce :

$$\begin{array}{l} R_2 \leftarrow R_2 - 2R_1 : \\ R_3 \leftarrow R_3 - R_1 : \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 3 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} R_3 \leftarrow R_3 + R_2 : \\ R_4 \leftarrow R_4 - R_3 : \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 \end{pmatrix}$$

Thus there are three non-zero rows, so the rank is 3. Hence the subspace has dimension 3.

Exercise 49. Let

$$U = \{(x, y, z) \in \mathbb{R}^3 : x = y\} \quad \text{and} \quad V = \{(x, y, z) \in \mathbb{R}^3 : y = z\}.$$

Find a basis for $U \cap V$ and $U + V$. Determine their dimensions.

Solution 49.

- U consists of vectors $\begin{pmatrix} x \\ x \\ z \end{pmatrix}$ with $x, z \in \mathbb{R}$. A basis for U is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Hence the subspace U has dimension 2
- V consists of vectors $\begin{pmatrix} x \\ y \\ y \end{pmatrix}$ with $x, y \in \mathbb{R}$. A basis for V is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. Hence the subspace V has dimension 2.
- Now $U \cap V$: vectors satisfying both $x = y$ and $y = z$, so $x = y = z$. Hence vectors of form $\begin{pmatrix} t \\ t \\ t \end{pmatrix}$, $t \in \mathbb{R}$. A basis is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, Hence the subspace $U \cap V$ has dimension 1.

— For $U + V$, we find the span of all vectors in U and V . Combine bases :

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

We can row reduce to find a basis. Form a matrix with these as rows :

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Row reduce : swap to get order, but we can see that these four vectors span \mathbb{R}^3 ?!!

Check : if they span all. The first three : $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ already span \mathbb{R}^3 ? For

instance, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ give $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ by subtraction, so we can get the standard basis.

Indeed, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ give two axes ; from $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ we get $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. So

$$U + V = \mathbb{R}^3 \quad \text{Hence the subspace } U + V \text{ has dimension 3.}$$

We can also verify dimension formula :

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V) = 2 + 2 - 1 = 3.$$

Exercise 50. In $\mathbb{R}_2[t]$, let

$$U = \{p(t) : p(0) = 0\} \quad \text{and} \quad V = \{p(t) : p'(0) = 0\}.$$

Find bases for U , V , $U \cap V$, and $U + V$.

Solution 50.

First, $\mathbb{R}_2[t]$ consists of polynomials $p(t) = a + bt + ct^2$.

— Let $p(t) \in \mathbb{R}_2[t]$, we have

$$p(t) \in U \Leftrightarrow a + bt + ct^2|_{t=0} = 0 \Leftrightarrow a = 0.$$

So

$$\forall p \in U \Leftrightarrow p(t) = bt + ct^2.$$

Thus $\{t, t^2\}$ is a basis for U : and $\dim U = 2$.

— $V = \{p : p'(0) = 0\}$. $p'(t) = b + 2ct$,
we have

$$p(t) \in V \Leftrightarrow (a + bt + ct^2)'|_{t=0} = 0 \Leftrightarrow (b + 2ct)|_{t=0} = 0 \Rightarrow b = 0.$$

So

$$\forall p \in V \Leftrightarrow p(t) = a + ct^2.$$

Thus $\{1, t^2\}$ is a basis for V : and $\dim V = 2$.

— $U \cap V$: polynomials with both $a = 0$ and $b = 0$, so

$$\forall p \in U \cap V \Leftrightarrow p(t) = ct^2.$$

Thus $\{t^2\}$ is a basis for $U \cap V$: and $\dim(U \cap V) = 1$.

— $U + V$: since U and V are subspaces of a 3-dimensional space, and

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V) = 2 + 2 - 1 = 3,$$

so

$$U + V = \mathbb{R}_2[t].$$

A basis is the standard basis $\{1, t, t^2\}$.

Exercise 51. Let W be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. Is W a subspace of $M_2(\mathbb{R})$?

Solution 51.

Check :

— Zero matrix has the form with $a = b = c = 0$, so $\mathbf{0} \in W$.

— Take two matrices $A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ c' & 0 \end{pmatrix}$. Their sum is

$$\begin{pmatrix} a + a' & b + b' \\ c + c' & 0 \end{pmatrix},$$

which is again of the required form. So closed under addition.

— Scalar multiplication :

$$\alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & 0 \end{pmatrix},$$

still of the form.

Thus W is a subspace. A basis could be

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

And the Dimension is 3.

Exercise 52. In the vector space $C[0, 1]$ of continuous functions on $[0, 1]$, consider the subset

$$S = \{f \in C[0, 1] : f(0) = f(1)\}.$$

| Show that S is a subspace and find its dimension (if infinite, state so).

Solution 52.

Check :

— Zero function satisfies $0(0) = 0(1)$, so $\mathbf{0} \in S$.

— If $f, g \in S$, then $f(0) = f(1)$ and $g(0) = g(1)$. Then

$$(f + g)(0) = f(0) + g(0) = f(1) + g(1) = (f + g)(1),$$

so $f + g \in S$.

— If $f \in S$ and $\alpha \in \mathbb{R}$, then

$$(\alpha f)(0) = \alpha f(0) = \alpha f(1) = (\alpha f)(1),$$

so $\alpha f \in S$.

Thus S is a subspace. The space is **infinite-dimensional**. For example,

the functions : $f(t) = \sin(2\pi nt)$, for $n \in \mathbb{N}$

satisfy $f(0) = f(1) = 0$ and are linearly independent, so dimension is infinite.

▷ In this section we write the vector in \mathbb{R}^n in component form as (x_1, x_2, \dots, x_n) .

Exercise 53. which of the following is linear map

$$\begin{aligned}
 T_1 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2, T_1(a, b, c) = (2a - b, a + c + 5), & T_2 : \mathbb{R}^3 &\longrightarrow \mathbb{R}, T_2(a, b, c) = 2a - b, \\
 T_3 : \mathbb{R} &\longrightarrow \mathbb{R}^3, T_3(a) = (a, 3a, \sin(a)), & T_4 : \mathcal{M}_2(\mathbb{R}) &\longrightarrow \mathbb{R}_2[x], T_4 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + c(x^2 + 1), \\
 T_5 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3, T_5(a, b, c) = (a + b, 0, c), & T_6 : \mathbb{R}_2[x] &\longrightarrow \mathbb{R}, T_6(a + bx + cx^2) = a - c + 3, \\
 T_7 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3, T_7(a, b, c) = (a^2 + b, 0, c), & T_8 : \mathbb{R}_2[x] &\longrightarrow \mathbb{R}_1[x], T_8(a + bx + cx^2) = a + b^2x, \\
 T_9 : \mathbb{R}^3 &\longrightarrow \mathbb{R}_2[x], T_9(a + bx + cx^2) = (a + b) + cx^2, & T_{10} : \mathbb{R}^2 &\longrightarrow \mathbb{R}, T_{10}(a, b) = |a|, \\
 T_{11} : \mathcal{M}_2(\mathbb{R}) &\longrightarrow \mathcal{M}_2(\mathbb{R}), T_{11}(A) = A^t A, & T_{12} : \mathcal{M}_2(\mathbb{R}) &\longrightarrow \mathcal{M}_2(\mathbb{R}), T_{12}(A) = A^2, \\
 T_{13} : \mathcal{M}_2(\mathbb{R}) &\longrightarrow \mathcal{M}_2(\mathbb{R}), T_{13}(A) = A^t + 2A, & T_{14} : \mathcal{M}_2(\mathbb{R}) &\longrightarrow \mathcal{M}_2(\mathbb{R}), T_{14}(A) = \det(A), \\
 T_{15} : \mathbb{R}_2[x] &\longrightarrow \mathbb{R}, T_{15}(P(x)) = P(2) - P(1), & T_{16} : \mathcal{F}(\mathbb{R}) &\longrightarrow \mathcal{F}(\mathbb{R}), T_{16}(f(x)) = xf(x), \\
 T_{17} : \mathcal{F}(\mathbb{R}) &\longrightarrow \mathcal{F}(\mathbb{R}), T_{17}(f(x)) = f(-x); & T_{18} : \mathcal{F}(\mathbb{R}) &\longrightarrow \mathcal{F}(\mathbb{R}), T_{18}(f(x)) = x^2 f(x)
 \end{aligned}$$

Solution 53. It easy to prove that :

- $T_1(a, b, c) = (2a - b, a + c + 5)$: not linear because $T_1(0, 0, 0) = (0, 5) \neq 0$.
- $T_2(a, b, c) = 2a - b$: linear.
- $T_3(a) = (a, 3a, \sin a)$: not linear due to $\sin a$.
- $T_4 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + c(x^2 + 1)$: linear.
- $T_5(a, b, c) = (a + b, 0, c)$: linear .
- $T_6(a + bx + cx^2) = a - c + 3$: not linear $T_6(0) = 3 \neq 0$.
- $T_7(a, b, c) = (a^2 + b, 0, c)$ not linear because of a^2 .
- $T_8(a + bx + cx^2) = a + b^2x$: not linear because of b^2 .
- $T_9(a + bx + cx^2) = (a + b) + cx^2$: linear.
- $T_{10}(a, b) = |a|$: not linear.
- $T_{11}(A) = A^t A$: not linear (quadratic).
- $T_{12}(A) = A^2$: not linear.
- $T_{13}(A) = A^t + 2A$: linear.
- $T_{14}(A) = \det A$: not linear.
- $T_{15}(P(x)) = P(2) - P(1)$: linear (difference of evaluation maps).
- $T_{16}(f(x)) = xf(x)$: linear on function space.
- $T_{17}(f(x)) = f(-x)$: linear.
- $T_{18}(f(x)) = x^2 f(x)$: linear.

Exercise 54. Determine whether each of the following maps $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear. If it is, prove it; if not, give a counterexample.

1. $T(x, y) = (x + y, 2x - y)$.
2. $T(x, y) = (x^2, y)$.

3. $T(x, y) = (x + 1, y)$.
4. $T(x, y) = (3x, -x + 2y)$.

Solution 54.

1. T is linear. **Check** : $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$$\begin{aligned} T((x_1, y_1) + (x_2, y_2)) &= T(x_1 + x_2, y_1 + y_2) \\ &= ((x_1 + x_2) + (y_1 + y_2), 2(x_1 + x_2) - (y_1 + y_2)) \\ &= (x_1 + y_1 + x_2 + y_2, 2x_1 - y_1 + 2x_2 - y_2) \\ &= T(x_1, y_1) + T(x_2, y_2). \end{aligned}$$

For scalar multiplication : $\forall (x, y) \in \mathbb{R}^2, \forall \alpha \in \mathbb{R}$,

$$T(\alpha(x, y)) = T(\alpha x, \alpha y) = (\alpha x + \alpha y, 2\alpha x - \alpha y) = \alpha(x + y, 2x - y) = \alpha T(x, y).$$

Hence linear.

2. T is not linear. For example, $T(2(1, 0)) = T(2, 0) = (4, 0)$ while

$$2T(1, 0) = 2(1, 0) = (2, 0) \neq (4, 0).$$

Or note that the first component is quadratic.

3. T is not linear because $T(0, 0) = (1, 0) \neq (0, 0)$. A linear map must send zero to zero.
4. T is linear. **Check** : $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$$\begin{aligned} T((x_1, y_1) + (x_2, y_2)) &= (3(x_1 + x_2), -(x_1 + x_2) + 2(y_1 + y_2)) \\ &= (3x_1 + 3x_2, -x_1 + 2y_1 - x_2 + 2y_2) \\ &= T(x_1, y_1) + T(x_2, y_2). \end{aligned}$$

Also $\forall (x, y) \in \mathbb{R}^2, \forall \alpha \in \mathbb{R}$,

$$T(\alpha x, \alpha y) = (3\alpha x, -\alpha x + 2\alpha y) = \alpha(3x, -x + 2y) = \alpha T(x, y).$$

Exercise 55. Let V be the vector space of all real-valued functions on \mathbb{R} . Determine whether the following operators $T : V \rightarrow V$ are linear.

1. $(Tf)(x) = f(x) + 1$.
2. $(Tf)(x) = f(x^2)$.
3. $(Tf)(x) = f'(x)$ (differentiation, assuming f is differentiable).
4. $(Tf)(x) = f(0)$ (constant function).

Solution 55.

1. Not linear because $T(0)(x) = 0(x) + 1 = 1 \neq 0$ (zero function).
2. Linear : $\forall f, g \in V$

$$(T(f + g))(x) = (f + g)(x^2) = f(x^2) + g(x^2) = (Tf)(x) + (Tg)(x),$$

and

$$\forall f \in V, \forall \alpha \in \mathbb{R}, \quad (T(\alpha f))(x) = (\alpha f)(x^2) = \alpha f(x^2) = \alpha(Tf)(x).$$

3. Differentiation is linear :

$$(f + g)' = f' + g', \quad \text{and} \quad (\alpha f)' = \alpha f'.$$

4. Linear :

$$(T(f + g))(x) = (f + g)(0) = f(0) + g(0) = (Tf)(x) + (Tg)(x);$$

similarly for scalar multiplication.

Exercise 56. For the linear map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + y, y + z, x - z),$$

find $\ker T$ and $\text{Im } T$. Determine their dimensions.

Solution 56.

Kernel :

$$\ker T = \{u = (x, y, z) \in \mathbb{R}^3, T(u) = 0_{\mathbb{R}^3}\}.$$

Solve $T(x, y, z) = (0, 0, 0)$:

$$\begin{cases} x + y = 0 \\ y + z = 0 \\ x - z = 0 \end{cases} \Rightarrow \begin{cases} x = x \\ y = -x \\ z = x \end{cases}.$$

So,

$$u \in \ker T \Leftrightarrow u = (x, -x, x) = x(1, -1, 1).$$

Hence

$$\ker T = \text{span}\{(1, -1, 1)\} \Rightarrow \dim(\ker) = 1.$$

Image : The image is spanned by the images of the standard basis vectors :

$$T(1, 0, 0) = (1, 0, 1), \quad T(0, 1, 0) = (1, 1, 0), \quad T(0, 0, 1) = (0, 1, -1).$$

We need a basis for these three vectors. Form a matrix with them as rows (or columns) and row reduce. Put them as rows :

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Row reduce :

$$R_2 \leftarrow R_2 - R_1 : \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow R_3 \leftarrow R_3 - R_2 : \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\text{rank} = 2$. Thus $\text{Im } T$ is 2-dimensional. A basis could be $\{(1, 0, 1), (0, 1, -1)\}$.

Check rank-nullity :

$$\dim \ker T + \dim \text{Im } T = 1 + 2 = 3.$$

Exercise 57. Let $T : \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x]$ be the linear transformation given by

$$T(p)(x) = p'(x) + p(0).$$

Find a basis for $\ker T$ and $\text{Im } T$.

Solution 57.

Write a general polynomial

$$p(x) = a + bx + cx^2.$$

Then

$$p'(x) = b + 2cx,$$

so

$$T(p)(x) = (b + 2cx) + a = (a + b) + 2cx.$$

Thus $T(p)$ is a linear polynomial (degree ≤ 1). In coordinates (a, b, c) (with respect to basis $\{1, x, x^2\}$), we have

$$T(a, b, c) = (a + b, 2c, 0), \text{ coefficients of } 1, x, x^2, \text{ respectively.}$$

Kernel : $T(p) = 0$ means $a + b = 0$ and $2c = 0$ and the coefficient of x^2 is already 0. So $c = 0$ and $b = -a$. Hence

$$p(x) = a - ax = a(1 - x).$$

Thus

$$\ker T = \text{span}\{1 - x\} \Rightarrow \dim(\ker(T)) = 1.$$

Image : The image consists of all polynomials of the form $(a + b) + 2cx$. That is, any linear polynomial $\alpha + \beta x$ can be obtained : take $c = \beta/2$ and choose a, b such that $a + b = \alpha$ (e.g., $a = 0, b = \alpha$). So

$$\text{Im } T = \{\alpha + \beta x : \alpha, \beta \in \mathbb{R}\},$$

which is the set of all polynomials of degree ≤ 1 . A basis is $\{1, x\}$ and $\dim(\text{Im}(T)) = 2$.

Check rank-nullity :

$$\dim(\ker T) + \dim(\text{Im } T) = 1 + 2 = 3 = \dim(\mathbb{R}_2[x]).$$

Exercise 58. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

Compute :

1. $A + B$
2. $3A - 2B$
3. AB and BA
4. A^T

Solution 58. 1. $A + B = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}.$

2. $3A = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}, 2B = \begin{pmatrix} 10 & 12 \\ 14 & 16 \end{pmatrix},$ so $3A - 2B = \begin{pmatrix} -7 & -6 \\ -5 & -4 \end{pmatrix}.$

$$3. AB = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

$$BA = \begin{pmatrix} 5 \cdot 1 + 6 \cdot 3 & 5 \cdot 2 + 6 \cdot 4 \\ 7 \cdot 1 + 8 \cdot 3 & 7 \cdot 2 + 8 \cdot 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}.$$

$$4. A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Exercise 59. Let $A \in \mathcal{M}_n$ be nilpotent of index p . Show that $I - A$ is invertible with

$$(I - A)^{-1} = I + A + A^2 + \dots + A^{p-1}.$$

Solution 59. By assumption, $A^p = 0$. Since

$$(I - A)(I + A + A^2 + \dots + A^{p-1}) = I + A + A^2 + \dots + A^{p-1} - A - A^2 - \dots - A^{p-1} - A^p, \\ = I,$$

we see that $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \dots + A^{p-1}.$$

Exercise 60. Let

$$A = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}.$$

By calculating A^2 , show that A is invertible and deduce the inverse matrix A^{-1} .

Solution 60. . We have

$$A^2 = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Since $A^2 = I_2$, we get $AA = I_2$, which clearly shows that A is invertible, and its inverse is itself.

Exercise 61. Let A be the matrix :

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

1. Check that $A^2 = 2I - A$, where I is the identity matrix of $\mathcal{M}_3(\mathbb{R})$.
2. Deduce that A is invertible and give A^{-1} .

Solution 61. 1. We have

$$A^2 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

On the other hand

$$2I - A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Thus

$$A^2 = 2I - A$$

2. We have :

$$A^2 = 2I - A \iff A^2 + A = 2I \iff \left(\frac{1}{2}(A + I)\right) A = A \left(\frac{1}{2}(A + I)\right) = I.$$

This shows that A is invertible, and its inverse is given by

$$\frac{1}{2}(A + I) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

Exercise 62. Suppose that the matrix A satisfies

$$(I + 2A)^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}.$$

Find A .

Solution 62. It is the inverse process! Set $B = (I + 2A)^{-1}$. Then, by the definition of an inverse matrix, we have :

$$B(I + 2A) = I_2.$$

Assume that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

hence

$$I + 2A = \begin{pmatrix} 2a + 1 & 2b \\ 2c & 2d + 1 \end{pmatrix}.$$

Let us determine the values of a , b , c and d satisfying the equation :

$$\begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2a + 1 & 2b \\ 2c & 2d + 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Expanding the left-hand side, we obtain :

$$\begin{pmatrix} -2a - 1 - 4c & -2b + 4d + 2 \\ 8a + 4 + 10c & 8b + 10d + 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solving the system, we find :

$$a = -9/13, \quad b = 1/13, \quad c = 2/13 \quad \text{and} \quad d = -6/13.$$

Thus, the sought matrix is :

$$A = \begin{pmatrix} -9/13 & 1/13 \\ 2/13 & -6/13 \end{pmatrix}$$

Exercise 63. Using the method of Gaussian elimination find the inverse of the matrix :

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{pmatrix}.$$

Solution 63. We have

$$\begin{aligned} & \left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 4 & -3 & 4 & 0 & 1 & 0 \\ 3 & -3 & 4 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 4 & -3 & 4 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 0 & 1 \end{array} \right) \xrightarrow{4R_3 - 3R_1 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 4 & -3 & 4 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 4 & 0 & -3 & 4 \end{array} \right) \\ & \xrightarrow{\begin{array}{l} R_1 + 3R_2 \rightarrow R_1 \\ R_3 + 2R_2 \rightarrow R_3 \end{array}} \left(\begin{array}{ccc|ccc} 4 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array} \right) \xrightarrow{\begin{array}{l} R_1 - R_3 \rightarrow R_1 \\ R_2 + R_3 \rightarrow R_2 \end{array}} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 0 & 4 & -4 \\ 0 & 1 & 0 & 4 & -3 & 4 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array} \right) \\ & \xrightarrow{\frac{1}{4}R_1 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 4 & -3 & 4 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array} \right) \end{aligned}$$

Thus

$$A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{pmatrix}$$

we notice that $A^{-1} = A$ and $A^2 = I_3$

Exercise 64. Using the method of Gaussian elimination find the inverse of the matrix :

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}.$$

Solution 64. We have

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & -8 & -9 \\ -1 & 2 & 5 & 0 & 8 & 9 \end{array} \right) \xrightarrow{\begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 + R_1 \rightarrow R_3 \end{array}} \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array} \right) \xrightarrow{R_2 + R_3 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 0 & -8 & -9 & -2 & 1 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & -1 & 1 & 1 \end{array} \right)$$

Since we have obtained a row of zeros on the left side, we conclude that A is not invertible

Exercise 65. Prove the nonexistence of square matrices of order $n \in \mathbb{N}^*$, A and B such that

$$AB - BA = I_n,$$

where I_n is the identity matrix of order n .

Solution 65. Assume, for the sake of contradiction, that such matrices A and B exist, that is, $AB - BA = I_n$, $n \neq 0$. Then,

$$\text{tr}(AB - BA) = \text{tr}(I_n) = 1 + 1 + \cdots + 1 = n,$$

and since

$$\text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0 \implies 0 = n,$$

which is absurd! Thus, no such matrices exist.

Exercise 66. Let $A \in \mathcal{M}_n(\mathbb{R})$. Assume that $\text{tr}(A^t A) = 0$. Show that $A = 0$.

Solution 66. Let $A = (a_{ij})$, where $1 \leq i, j \leq n$, then the diagonal elements of $A^t A$ are of the form

$$a_{ii} = \sum_{k=1}^n a_{ik}^2$$

Since $\text{tr}(A^t A) = 0$, we must have

$$\sum_{i=1}^n a_{ii}^2 = 0$$

Since each a_{ii} is nonnegative, we have

$$a_{11} = a_{22} = \dots = a_{nn} = 0.$$

Thus

$$\sum_{k=1}^n a_{ik}^2 = 0, \text{ for all } i$$

, which due to the nonnegativity of a_{ik}^2 , then implies that $a_{ij} = 0$ for all $1 \leq i, j \leq n$. Consequently, $A = 0$, as required.

Remark 5.0.1. Remark. If $A \in \mathcal{M}_n(\mathbb{C})$, (has complex entries), then $\text{tr}(A^t A) = 0$ does not necessarily yield $A = 0$. A counterexample is

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

where it can be shown that

$$\text{tr}(A^t A) = \text{tr} \left[\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \right] = \text{tr} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0, \text{ yet } A \neq 0.$$

Exercise 67. Define

$$T : M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R}) \text{ by } T(A) = A^T \text{ (transpose).}$$

Find $\ker T$ and $\text{Im } T$. Is T injective? Surjective?

Solution 67.

Kernel : $T(A) = 0$ means $A^T = 0$, hence $A = 0$. So

$$\ker T = \{0\} \Rightarrow \dim(\ker T) = 0 \Rightarrow T \text{ is injective.}$$

Image : Since $T(A) = A^T$, for any matrix B , $B = T(B^T)$, so every matrix is in the image. Hence

$$\text{Im } T = M_2(\mathbb{R}) \Rightarrow \dim(\text{Im } T) = 4 \Rightarrow T \text{ is surjective.}$$

In fact, T is an isomorphism because $T^2 = I$.

Exercise 68. Find all square matrices of order 2. X such that $X^2 = I$ (the identity matrix). Such matrices are called involutory.

Solution 68. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$X^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives the system :

$$\begin{cases} a^2 + bc = 1 & (1) \\ ab + bd = b(a + d) = 0 & (2) \\ ac + cd = c(a + d) = 0 & (3) \\ bc + d^2 = 1 & (4) \end{cases}$$

We have either $a + d = 0$ or $b = c = 0$.

Case 1 : $a + d = 0$, i.e. $d = -a$. Then (1) and (4) become

$$a^2 + bc = 1 \quad \text{and} \quad bc + a^2 = 1 \quad (\text{same}).$$

So $a^2 + bc = 1$. Thus any matrix of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $bc = 1 - a^2$ satisfies $X^2 = I$.

This includes, for example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

Case 2 : $b = c = 0$. Then (1) gives $a^2 = 1$, (4) gives $d^2 = 1$, so $a = \pm 1$, $d = \pm 1$. Thus

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

But note that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$ also satisfies $(-I)^2 = I$. So these are the diagonal involutory matrices.

Thus the general solution is : either X is diagonal with ± 1 on diagonal, or X has zero trace and $a^2 + bc = 1$.

Exercise 69. Compute the determinants of the following matrices.

$$A = \begin{pmatrix} 2 & 5 \\ 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution 69.

— $\det A = (2)(-1) - (5)(3) = -2 - 15 = -17$.

— For B , expand along first row :

$$\det B = 1 \cdot \det \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} = -15.$$

— C is upper triangular, so its determinant is the product of diagonal entries : $1 \cdot 1 \cdot 1 \cdot 1 = 1$.

Exercise 70. Use row operations to compute $\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$.

Solution 70. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$. Perform :

$$\begin{aligned} R_2 \leftarrow R_2 - 2R_1 : & \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 3 & 4 & 5 \end{pmatrix} \\ R_3 \leftarrow R_3 - 3R_1 : & \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix} \\ R_3 \leftarrow R_3 - 2R_2 : & \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The last row is zero, so $\det A = 0$. (Alternatively, note that rows are linearly dependent : $R_1 + R_3 = 2R_2$.)

Exercise 71. Let A and B be square matrices of order n . Prove that

$$\det(AB) = \det A \cdot \det B.$$

Solution 71.

This is a standard theorem. One proof : For an elementary matrix E ,

$$\det(EA) = \det E \cdot \det A.$$

Since any invertible matrix is a product of elementary matrices, and if A is invertible we can write

$$A = E_1 \cdots E_k,$$

then

$$\det(AB) = \det(E_1 \cdots E_k B) = \det E_1 \cdots \det E_k \det B = \det A \det B.$$

For non-invertible A , both sides are zero because $\det A = 0$ and $\det(AB) = 0$ (since AB is singular). The case where A invertible but B singular is similar.

Exercise 72. Show that $\det(A^T) = \det A$.

Solution 72.

This can be proved by induction using the definition of determinant via cofactor expansion. **Alternatively**, note that row operations correspond to column operations on the transpose, and the effect on determinant is the same. Formally, one can show that the determinant of a matrix and its transpose are equal because they have the same characteristic polynomial.

Exercise 73. Let A be an square matrix of order n , such that $A^2 = A$ (idempotent). Show that $\det A$ is either 0 or 1.

Solution 73.

Take determinants :

$$\det(A^2) = \det A \cdot \det A = (\det A)^2.$$

But $\det(A^2) = \det A$. So

$$(\det A)^2 = \det A,$$

hence

$$\det A(\det A - 1) = 0.$$

Therefore

$$\det A = 0 \quad \text{or} \quad \det A = 1.$$

Exercise 74. Find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ using the adjugate formula.

Solution 74.

$$\det A = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2.$$

The adjugate is the transpose of the cofactor matrix. Cofactors :

$$C_{11} = 4, \quad C_{12} = -3, \quad C_{21} = -2, \quad C_{22} = 1.$$

So the cofactor matrix is

$$\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}, \quad \text{its transpose is} \quad \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Thus

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Check :

$$AA^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} = \begin{pmatrix} 1(-2) + 2(1.5) & 1(1) + 2(-0.5) \\ 3(-2) + 4(1.5) & 3(1) + 4(-0.5) \end{pmatrix} = \begin{pmatrix} -2 + 3 & 1 - 1 \\ -6 + 6 & 3 - 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 75. For the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{pmatrix},$$

compute $\text{adj}(A)$ and verify that $A \text{adj}(A) = (\det A)I$.

Solution 75.

First compute $\det A$. Expand along first row :

$$\det A = 1 \cdot \det \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = 11.$$

Now compute cofactors :

$$C_{11} = + \det \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} = (-1)(0) - 1(3) = -3,$$

$$C_{12} = - \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -(2 \cdot 0 - 1 \cdot 1) = -(0 - 1) = 1,$$

$$C_{13} = + \det \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = (2)(3) - (-1)(1) = 6 + 1 = 7,$$

$$C_{21} = - \det \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = -(0 \cdot 0 - 2 \cdot 3) = -(-6) = 6,$$

$$C_{22} = + \det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = (1)(0) - (2)(1) = -2,$$

$$C_{23} = - \det \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} = -(1 \cdot 3 - 0 \cdot 1) = -3,$$

$$C_{31} = + \det \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} = (0)(1) - (2)(-1) = 2,$$

$$C_{32} = - \det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -(1 \cdot 1 - 2 \cdot 2) = -(1 - 4) = 3,$$

$$C_{33} = + \det \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = (1)(-1) - (0)(2) = -1.$$

Thus the cofactor matrix is

$$\text{cof}(A) = \begin{pmatrix} -3 & 1 & 7 \\ 6 & -2 & -3 \\ 2 & 3 & -1 \end{pmatrix}.$$

The adjugate is its transpose :

$$\text{adj}(A) = \begin{pmatrix} -3 & 6 & 2 \\ 1 & -2 & 3 \\ 7 & -3 & -1 \end{pmatrix}.$$

Now compute $A \text{adj}(A)$:

$$\begin{aligned} A \text{adj}(A) &= \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -3 & 6 & 2 \\ 1 & -2 & 3 \\ 7 & -3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1(-3) + 0 \cdot 1 + 2(7) & 1 \cdot 6 + 0(-2) + 2(-3) & 1 \cdot 2 + 0 \cdot 3 + 2(-1) \\ 2(-3) + (-1)1 + 1(7) & 2 \cdot 6 + (-1)(-2) + 1(-3) & 2 \cdot 2 + (-1)3 + 1(-1) \\ 1(-3) + 3 \cdot 1 + 0(7) & 1 \cdot 6 + 3(-2) + 0(-3) & 1 \cdot 2 + 3 \cdot 3 + 0(-1) \end{pmatrix} \\ &= \begin{pmatrix} -3 + 0 + 14 & 6 + 0 - 6 & 2 + 0 - 2 \\ -6 - 1 + 7 & 12 + 2 - 3 & 4 - 3 - 1 \\ -3 + 3 + 0 & 6 - 6 + 0 & 2 + 9 + 0 \end{pmatrix} = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix} = 11I = (\det A)I. \end{aligned}$$

Exercise 76. Use Cramer's rule to solve the system

$$\begin{cases} 2x + 3y = 8 \\ 4x - y = 2 \end{cases}$$

Solution 76. Write in matrix form

$$\mathbf{Ax} = \mathbf{b} \quad \text{with} \quad A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}.$$

Compute

$$\det A = 2(-1) - 3(4) = -2 - 12 = -14.$$

For x , replace first column with \mathbf{b} :

$$A_x = \begin{pmatrix} 8 & 3 \\ 2 & -1 \end{pmatrix}, \quad \det A_x = 8(-1) - 3(2) = -8 - 6 = -14,$$

Thus

$$x = \frac{\det A_x}{\det A} = \frac{-14}{-14} = 1.$$

For y , replace second column :

$$A_y = \begin{pmatrix} 2 & 8 \\ 4 & 2 \end{pmatrix}, \quad \det A_y = 2 \cdot 2 - 8 \cdot 4 = 4 - 32 = -28.$$

Thus

$$y = \frac{\det A_y}{\det A} = \frac{-28}{-14} = 2.$$

Therefore $(x, y) = (1, 2)$.

Exercise 77. Use Cramer's rule to find z in the system

$$\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 7 \\ 3x + 2y - z = 4 \end{cases}$$

Solution 77. Write in matrix form

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 7 \\ 4 \end{pmatrix}$$

. First Compute

$$\det A = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix} = 13.$$

For x , replace first column with \mathbf{b} :

$$A_x = \begin{pmatrix} 6 & 1 & 1 \\ 7 & -1 & 3 \\ 4 & 2 & -1 \end{pmatrix}, \quad \det A_x = 7,$$

Thus

$$x = \frac{\det A_x}{\det A} = \frac{7}{13}.$$

For y , replace second column with \mathbf{b} :

$$A_y = \begin{pmatrix} 1 & 6 & 1 \\ 2 & 7 & 3 \\ 3 & 4 & -1 \end{pmatrix}, \quad \det A_y = 34.$$

Thus

$$y = \frac{\det A_y}{\det A} = \frac{34}{13}.$$

Now for z , replace third column with \mathbf{b} :

$$A_z = \begin{pmatrix} 1 & 1 & 6 \\ 2 & -1 & 7 \\ 3 & 2 & 4 \end{pmatrix}. \quad \det A_z = 37$$

Thus

$$z = \frac{\det A_z}{\det A} = \frac{37}{13}.$$

Therefore $(x, y, z) = (7/13, 34/13, 37/13)$.

Exercise 78. Show that the determinant of an orthogonal matrix is ± 1 .

Solution 78. If Q is orthogonal, then $Q^T Q = I$. Taking determinants,

$$\det(Q^T Q) = \det Q^T \det Q = (\det Q)^2 = \det I = 1, \quad \text{so} \quad \det Q = \pm 1.$$

Exercise 79. Let A be a skew-symmetric matrix of odd order. Prove that $\det A = 0$.

Solution 79.

If A is skew-symmetric, $A^T = -A$. Taking determinants,

$$\det(A^T) = \det(-A) = (-1)^n \det A.$$

But $\det(A^T) = \det A$. Hence

$$\det A = (-1)^n \det A.$$

If n is odd, $(-1)^n = -1$, so $\det A = -\det A$, implying $\det A = 0$.

Exercise 80. Find the area of the parallelogram in \mathbb{R}^2 spanned by the vectors

$$u = (2, 1) \quad \text{and} \quad v = (3, 4).$$

Solution 80. The area is the absolute value of the determinant of the matrix whose columns (or rows) are \mathbf{u} and \mathbf{v} :

$$\det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = 5. \quad \text{Thus area} = 5.$$

Exercise 81. Find the volume of the parallelepiped in \mathbb{R}^3 spanned by

$$u = (1, 0, 1), \quad v = (2, 1, 0), \quad w = (0, 1, 2).$$

Solution 81. The volume is given by

$$V = |\det[uv, w]|.$$

Compute :

$$\det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 4. \quad \text{Thus volume} = 4.$$

Exercise 82. Let A be a square matrix of order n , with integer entries. Prove that A^{-1} has integer entries if and only if $\det A = \pm 1$.

Solution 82.

— If A^{-1} has integer entries, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

must be an integer (since determinant of an integer matrix is an integer). Thus $\frac{1}{\det A}$ is integer, so $\det A = \pm 1$.

— **Conversely** : If $\det A = \pm 1$, then

$$A^{-1} = \frac{1}{\det A} \text{adj}(A).$$

The adjugate of an integer matrix has integer entries, so A^{-1} has integer entries.

Exercise 83. Show that for any square matrices of order n , A and B ,

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A + B) \det(A - B).$$

Solution 83. Use block matrix operations :

$$\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} = \begin{pmatrix} A + B & 0 \\ B & A - B \end{pmatrix}.$$

The left and right multipliers have determinant 1, so the determinant of the original block matrix equals $\det(A + B) \det(A - B)$ (since the determinant of a block triangular matrix is the product of the determinants of the diagonal blocks).

Exercise 84. Prove that for any square matrix of order n A ,

$$\det(\text{adj}(A)) = (\det A)^{n-1}.$$

Solution 84. We know that $A \text{adj}(A) = (\det A)I$. Taking determinants :

$$\det(A) \det(\text{adj}(A)) = \det((\det A)I) = (\det A)^n.$$

— If $\det A \neq 0$, we can divide to get

$$\det(\operatorname{adj}(A)) = (\det A)^{n-1}.$$

— If $\det A = 0$, then $\operatorname{adj}(A)$ is singular (unless $n = 1$), and the formula still holds as $0 = 0^{n-1}$ (with 0^0 interpreted as 1 for $n = 1$). For $n = 1$, $\operatorname{adj}(A) = 1$ and $\det A = a$, so $\det(\operatorname{adj}(A)) = 1 = a^0$ holds.

Exercise 85. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by

$$T(x, y, z) = (2x - y, x + 3z).$$

Find the matrix of T with respect to the standard bases.

Solution 85. Compute images of standard basis vectors of \mathbb{R}^3 :

$$T(1, 0, 0) = (2, 1), \quad T(0, 1, 0) = (-1, 0), \quad T(0, 0, 1) = (0, 3).$$

Place these as columns of a 2×3 matrix :

$$[T] = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

Check : For $v = (x, y, z)$,

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + 3z \end{pmatrix}.$$

Exercise 86. Let $T : \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x]$ be given by

$$T[p(x)] = p(x) + p'(x).$$

Find the matrix of T with respect to the standard basis $\{1, x, x^2\}$.

Solution 86. Let $p(x) = a + bx + cx^2$. Then $p'(x) = b + 2cx$, so

$$T[p(x)] = (a + b) + (b + 2c)x + cx^2.$$

In coordinates (a, b, c) , we have

$$T(a, b, c) = (a + b, b + 2c, c).$$

Thus the matrix is :

$$[T] = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(columns are images of basis vectors).

Check :

$$T[1] = 1 \text{ gives } (1, 0, 0)$$

$$T[x] = 1 + x \text{ gives } (1, 1, 0)$$

$$T[x^2] = 2x + x^2 \text{ gives } (0, 2, 1)$$

Exercise 87. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map defined by

$$T(X) = AX \quad \text{where} \quad A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & -4 \\ 2 & 4 & 6 \end{pmatrix}.$$

1. Find $T \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Is the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ in $\ker(T)$? Is the vector $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ in $\text{Im}(T)$?
2. Find basis for $\text{Im}(T)$
3. Find basis for $\ker(T)$
4. Find $\text{rank}(T)$ and $\text{Nullity}(T)$.

Solution 87.

1.

$$T \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + (-2) \cdot 0 + 3 \cdot 2 \\ -1 \cdot 1 + 3 \cdot 0 + (-4) \cdot 2 \\ 2 \cdot 1 + 4 \cdot 0 + 6 \cdot 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ 14 \end{pmatrix}.$$

- Check if $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in \ker T$: compute

$$A^t(1, 2, 1) = (1 - 4 + 3 = 0, -1 + 6 - 4 = 1, 2 + 8 + 6 = 16) \neq 0, \quad \text{so no.}$$

- Check if ${}^t(1, 1, 3) \in \text{Im } T$: Solve $AX = {}^t(1, 1, 3)$. Augmented matrix :

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -1 & 3 & -4 & 1 \\ 2 & 4 & 6 & 3 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 8 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 8 & -15 \end{array} \right)$$

The system is consistent (last row gives $8z = -15$), so yes, it is in the image.

2. Find kernel : solve $AX = 0$. Row reduce A :

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & -4 \\ 2 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 8 \end{pmatrix}$$

So $\text{rank} = 3$ (since last row gives $8z = 0 \Rightarrow z = 0$, then $y = 0$, $x = 0$). Thus

$$\ker T = \{0\}.$$

-

3. $\text{Nullity} = 0$. Image is all of \mathbb{R}^3 , rank 3.

4. Basis for image : standard basis ; basis for kernel : none (empty set).

Exercise 88. Write the matrix associated with the linear map $f : \mathcal{M}_2(\mathbb{R}) \rightarrow \mathbb{R}$ defined by $f(A) = \text{tr} A$, with respect to the canonical bases.

Solution 88. Denote the matrix associated with f , which is of size $\dim(\mathbb{R}) \times \dim(\mathcal{M}_2(\mathbb{R})) = 1 \times 2^2$, by $[f]$. Recall that the standard basis of $\mathcal{M}_2(\mathbb{R})$ is given by

$$E_{11} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Since

$$\operatorname{tr}E_{11} = \operatorname{tr}E_{22} = 1 \quad \text{and} \quad \operatorname{tr}E_{12} = \operatorname{tr}E_{21} = 0,$$

the matrix $[f]$ is given by :

$$[f] = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 89. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a rotation by $\pi/3$ counterclockwise. Find its matrix with respect to the standard basis. Then find the matrix of T with respect to the basis $B = \{(1, 1), (1, -1)\}$.

Solution 89.

Rotation by θ has matrix

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The matrix of T with respect to the standard basis

$$[T]_{std} = \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Now for basis $B = \{v_1, v_2\}$ with $v_1 = (1, 1)$, $v_2 = (1, -1)$. First find the change-of-basis matrix P from B to standard :

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $[T]_B = P^{-1}[T]_{std}P$. Compute

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

(since $\det P = -2$) ? Actually

$$\det P = (1)(-1) - (1)(1) = -2,$$

so

$$P^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

So

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(note this equals $\frac{1}{2}P^T$ but not exactly). Then compute :

$$[T]_B = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Multiply stepwise : Let $M = [T]_{std}P$. First compute M :

$$M = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} + \frac{1}{2} & \frac{\sqrt{3}}{2} - \frac{1}{2} \end{pmatrix}.$$

Then multiply by P^{-1} on the left :

$$[T]_B = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} M = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} + \frac{1}{2} & \frac{\sqrt{3}}{2} - \frac{1}{2} \end{pmatrix}.$$

Thus

$$[T]_B = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

This is a rotation matrix as well (by $\pi/3$ but maybe in a different coordinate system).

Exercise 90. Construct a linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ with $\ker T = \text{Im} T$. Is this possible ?

Solution 90. We know that

$$\ker T = \{v \in \mathbb{R}^4 \mid T(v) = 0\}, \quad \text{Im} T = \{f(v) \in \mathbb{R}^4 \mid v \in \mathbb{R}^4\}$$

If $\ker T = \text{Im} T$, then any vector in the image is also in the kernel, so T applied to any vector in the image gives zero. In particular, for any v ,

$$T(T(v)) = 0 \implies T^2 = 0.$$

Also, by rank-nullity,

$$\text{if } \dim(\ker T) = d \text{ then } \dim(\text{Im} T) = 4 - d.$$

Setting them equal gives

$$d = 4 - d \implies d = 2.$$

So we need a linear map on \mathbb{R}^4 with

$$\dim \ker T = 2, \quad \dim \text{Im} T = 2, \quad \text{and} \quad \text{Im} T \subseteq \ker T.$$

so

$$T^2 = 0.$$

This is possible. **For example**, define T on standard basis :

$$T(e_1) = e_1, \quad T(e_2) = e_2, \quad T(e_3) = 0, \quad T(e_4) = 0$$

would give image spanned by e_1, e_2 but kernel is spanned by e_3, e_4 and e_1, e_2 are not in kernel. So that doesn't satisfy $\text{Im} T \subseteq \ker T$. We need image contained in kernel. So choose :

$$T(e_1) = e_3, \quad T(e_2) = e_4, \quad T(e_3) = 0, \quad T(e_4) = 0.$$

Then

$$\text{Im} T = \text{span}\{e_3, e_4\}, \quad \ker T = \text{span}\{e_3, e_4\}.$$

Actually **check** :

$$T(e_3) = 0, \quad T(e_4) = 0 \quad \text{so} \quad e_3, e_4 \in \ker T.$$

Also $T(e_1) = e_3$ not zero, so $e_1 \notin \ker T$. Kernel consists of vectors

$$\ker T = \{\alpha e_1 + \beta e_2 + \gamma e_3 + \lambda e_4, \alpha, \beta, \gamma, \lambda \in \mathbb{R}\},$$

such that

$$T(v) = \alpha e_3 + \beta e_4 = 0 \Rightarrow \alpha = \beta = 0.$$

So

$$\ker T = \text{span}\{e_3, e_4\}.$$

That equals image. So this works.

Exercise 91. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear map given by the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of T .

Solution 91. Characteristic polynomial :

$$\det(A - \lambda I) = \det \begin{pmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{pmatrix} = (3 - \lambda)^2 - 1 = (\lambda - 2)(\lambda - 4).$$

Eigenvalues : $\lambda_1 = 2, \lambda_2 = 4$.

— For $\lambda = 2$: solve $(A - 2I)v = 0$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x + y = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ first eigenvector.}$$

— For $\lambda = 4$: $(A - 4I)v = 0$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow -x + y = 0 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ second eigenvector}$$

Exercise 92. Let $T : \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x]$ be defined by

$$T[p(x)] = p(x) + xp'(x).$$

Find the eigenvalues of T and a basis of eigenvectors.

Solution 92.

Write

$$p(x) = a + bx + cx^2. \quad \text{Then } p'(x) = b + 2cx,$$

so

$$\begin{aligned} T[p(x)] &= a + bx + cx^2 + x(b + 2cx) \\ &= a + bx + cx^2 + bx + 2cx^2 \\ &= a + 2bx + 3cx^2. \end{aligned}$$

Thus in coordinates (a, b, c) ,

$$T(a, b, c) = (a, 2b, 3c)$$

which gives

$$T(1, 0, 0) = (1, 0, 0), \quad T(0, 1, 0) = 2(0, 1, 0) \text{ and } T(0, 0, 1) = 3(0, 0, 1)$$

Or

$$T(1) = 1, \quad T(x) = 2x \quad \text{and} \quad T(x^2) = 3x^2.$$

So eigenvalues : 1, 2, 3 corresponding eigenvectors 1, x , x^2 .

Exercise 93. Let $S, T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be linear maps given by

$$S(x, y, z) = (x, y, 0), \quad T(x, y, z) = (x, z, y).$$

Find the matrices of S , T , $S \circ T$, and $T \circ S$. Verify that $S \circ T \neq T \circ S$.

Solution 93.

Standard matrices (with respect to standard basis) :

$$[S] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$[S \circ T] = [S][T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$[T \circ S] = [T][S] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Clearly different, so $S \circ T \neq T \circ S$.

Exercise 94. Show that the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$T(x, y) = (2x + y, x + y)$$

is invertible, and find T^{-1} .

Solution 94. The matrix of T is given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Its determinant $\det A = 1 \neq 0$, so A is invertible. The inverse matrix is

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Thus T^{-1} is given by

$$T^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{and} \quad T^{-1}(x, y) = (x - y, -x + 2y).$$

Exercise 95. Let V be a finite-dimensional vector space and let $T : V \rightarrow V$ be linear. Prove that

$$\text{if } T^2 = T \text{ (a projection), then } V = \ker T \oplus \text{Im} T.$$

Solution 95.

— First, note that if $v \in \text{Im} T \cap \ker T$, then $v = T(u)$ for some u and also $T(v) = 0$. Then

$$0 = T(v) = T(T(u)) = T^2(u) = T(u) = v.$$

So the intersection is

$$\text{Im} T \cap \ker T = \{0\}.$$

— For any $v \in V$, write $v = T(v) + (v - T(v))$. Clearly

$$T(v) \in \text{Im} T.$$

Also

$$T(v - T(v)) = T(v) - T^2(v) = T(v) - T(v) = 0,$$

so

$$v - T(v) \in \ker T.$$

Hence

$$V = \text{Im} T + \ker T.$$

Since the sum is direct ($\text{Im} T \cap \ker T = \{0\}$ intersection trivial), we have

$$V = \text{Im} T \oplus \ker T.$$

Exercise 96. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear map that reflects vectors across the plane $x + y + z = 0$. Find the matrix of T (with respect to the standard basis).

Solution 96. The reflection across a plane through the origin with unit normal \mathbf{n} is given by

$$T(\mathbf{v}) = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n}.$$

For the plane $x + y + z = 0$, a normal vector is $\mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1)$. Then

$$T(\mathbf{v}) = \mathbf{v} - 2 \left(\frac{v_1 + v_2 + v_3}{\sqrt{3}} \right) \frac{1}{\sqrt{3}}(1, 1, 1) = \mathbf{v} - \frac{2}{3}(v_1 + v_2 + v_3)(1, 1, 1).$$

Thus in coordinates,

$$T(x, y, z) = \left(x - \frac{2}{3}(x + y + z), y - \frac{2}{3}(x + y + z), z - \frac{2}{3}(x + y + z) \right).$$

Simplify :

$$\begin{cases} x' = \frac{1}{3}x - \frac{2}{3}y - \frac{2}{3}z \\ y' = -\frac{2}{3}x + \frac{1}{3}y - \frac{2}{3}z \\ z' = -\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z \end{cases}$$

Hence the matrix is

$$\frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.$$

Exercise 97. Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(x, y) = (x + y, 2x - y).$$

| Show that T is linear and find its matrix with respect to the standard basis.

Solution 97. Check Linearity :

— $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 :$

$$\begin{aligned} T((x_1, y_1) + (x_2, y_2)) &= T(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + y_1 + y_2, 2(x_1 + x_2) - (y_1 + y_2)) \\ &= (x_1 + y_1 + x_2 + y_2, 2x_1 - y_1 + 2x_2 - y_2) = T(x_1, y_1) + T(x_2, y_2). \end{aligned}$$

— **For scalar multiplication :** $\forall (x, y) \in \mathbb{R}^2, \forall \alpha \in \mathbb{R} :$

$$T(\alpha(x, y)) = T(\alpha x, \alpha y) = (\alpha x + \alpha y, 2\alpha x - \alpha y) = \alpha(x + y, 2x - y) = \alpha T(x, y).$$

Thus T is linear. The matrix with respect to standard basis is formed by the images of $(1, 0)$ and $(0, 1)$:

$$T(1, 0) = (1, 2), \quad T(0, 1) = (1, -1) \quad \Rightarrow \quad [T] = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

| **Exercise 98.** Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T(x, y, z) = (x + y, y + z).$$

Find the kernel and image of T . Determine their dimensions.

Solution 98.

1. **Kernel :**

$$\ker T = \{(x, y, z) \mid x + y = 0 \text{ and } y + z = 0\}.$$

From $x + y = 0$ we have $x = -y$; from $y + z = 0$ we have $z = -y$. So

$$(x, y, z) = (-y, y, -y) = y(-1, 1, -1).$$

Thus

$$\ker T = \text{span}\{(-1, 1, -1)\}, \text{ a one-dimensional subspace.}$$

2. **Image :** The image is the span of $T(e_1), T(e_2), T(e_3)$:

$$T(1, 0, 0) = (1, 0), \quad T(0, 1, 0) = (1, 1), \quad T(0, 0, 1) = (0, 1).$$

These three vectors span \mathbb{R}^2 since $(1, 0)$ and $(0, 1)$ can be obtained as combinations.
Hence

$$\text{Im} T = \mathbb{R}^2, \text{ dimension } 2.$$

The rank-nullity theorem :

$$\dim \ker T + \dim \text{Im} T = 1 + 2 = 3 = \dim \mathbb{R}^3, \text{ satisfied.}$$

| **Exercise 99.** Let $T_i : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, where

$$T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ x - z \end{pmatrix}, \quad T_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ x - 1 \end{pmatrix}.$$

Determine whether T_i are linear or not

Solution 99. Let $\alpha \in \mathbb{R}$, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in \mathbb{R}^3$, we have

1.

$$\begin{aligned} T_1(\alpha u + v) &= T_1 \left(\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \right) = T_1 \begin{pmatrix} \alpha x + x' \\ \alpha y + y' \\ \alpha z + z' \end{pmatrix} \\ &= \begin{pmatrix} (\alpha x + x') + (\alpha y + y') \\ (\alpha x + x') - (\alpha z + z') \end{pmatrix} = \begin{pmatrix} \alpha(x + y) \\ \alpha(x - z) \end{pmatrix} + \begin{pmatrix} x' + y' \\ x' - z' \end{pmatrix} \\ &= \alpha \begin{pmatrix} x + y \\ x - z \end{pmatrix} + \begin{pmatrix} x' + y' \\ x' - z' \end{pmatrix} = \alpha T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} + T_1 \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\ &= \alpha T_1(u) + T_1(v) \end{aligned}$$

thus T_1 is linear map.

2. We can see that $T_2 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so T_2 is not linear.

Exercise 100. Let $T : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathbb{R}^2$, where

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - b \\ c - d \end{pmatrix}.$$

Determine whether T is linear or not

Solution 100. Let $\alpha \in \mathbb{R}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$, we have

$$\begin{aligned} T(\alpha A + B) &= T \left(\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = T \begin{pmatrix} \alpha a + a' & \alpha b + b' \\ \alpha c + c' & \alpha d + d' \end{pmatrix} \\ &= \begin{pmatrix} \alpha(a - b) + a' - b' \\ \alpha(c - d) + c' - d' \end{pmatrix} = \alpha \begin{pmatrix} a - b \\ c - d \end{pmatrix} + \begin{pmatrix} a' - b' \\ c' - d' \end{pmatrix} \\ &= \alpha T(A) + T(B) \end{aligned}$$

thus T is linear map.

Exercise 101. Let $T : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathbb{R}$, where $T(A) = \text{tr}(A)$. Determine whether T is linear or not

Solution 101. Let $\alpha \in \mathbb{R}$ and $A, B \in \mathcal{M}_2(\mathbb{R})$. Let $A = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. So

$$\begin{aligned} T(\alpha A + B) &= T\left(\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = T\begin{pmatrix} \alpha a + e & \alpha b + f \\ \alpha c + g & \alpha d + h \end{pmatrix} \\ &= \text{tr}\begin{pmatrix} \alpha a + e & \alpha b + f \\ \alpha c + g & \alpha d + h \end{pmatrix} = \alpha a + e + \alpha d + h \\ &= \alpha(a + d) + e + h = \alpha \text{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \text{tr}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \alpha \text{tr}(A) + \text{tr}(B) = \alpha T(A) + T(B) \end{aligned}$$

thus T is linear map.

Exercise 102. Let $f : \mathbb{R} \rightarrow \mathbb{R}^2, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be two linear mappings defined by :

$$f(x) = (-x, x) \quad \text{and} \quad g(x, y) = (x + y, x - 3y) \quad \text{respectively.}$$

1. Find $g \circ f$.
2. Find the matrices associated with f, g and $g \circ f$ with respect to their canonical bases.

Solution 102.

1. The composition $g \circ f$ is defined from \mathbb{R} to \mathbb{R}^2 by :

$$(g \circ f)(x) = g(f(x)) = g(-x, x) = (0, -4x).$$

2. The canonical basis of \mathbb{R} is 1. Then, we have

$$f(1) = (-1, 1),$$

which signifies that the associated matrix is given by :

$$[f] = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

As for g , we have $g(1, 0) = (1, 1)$ and $g(0, 1) = (1, -3)$, and so

$$[g] = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}.$$

To find the associated matrix $[g \circ f]$, we will present two methods to achieve this.

First method : Since $(g \circ f)(1) = (0, -4)$, we see that

$$[g \circ f] = \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

Second method : We know that $[g \circ f] = [g][f]$, and so

$$[g \circ f] = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

Exercise 103. Let $E = \mathbb{R}_3[x]$ and let $f : \mathbb{R}^2 \rightarrow E$ be a linear map defined by :

$$f(a, b) = a + b + ax + bx^2 + (a - b)x^3.$$

1. Determine the matrix associated with f , noted A , with respect to the canonical bases of \mathbb{R}^2 and E .
2. Let $g : E \rightarrow \mathbb{R}^2$ be defined by :

$$g[P(X)] = (P(0), P(1)).$$

Say why $g \circ f$ is linear and give its matrix in the canonical basis of \mathbb{R}^2

Solution 103.

1. Since $\dim E = 4$ and $\dim \mathbb{R}^2 = 2$, matrix A is of size 4×2 . We have

$$f(1, 0) = 1 + x + x^3 = 1 \times 1 + 1 \times x + 0 \times x^2 + 1 \times x^3$$

and

$$f(0, 1) = 1 + x^2 - x^3 = 1 \times 1 + 0 \times x + 1 \times x^2 + (-1) \times x^3.$$

Therefore,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

2. We can easily show that g is linear, then $g \circ f$ be linear (since f is already linear). To find the matrix C associated with $g \circ f$, we can either multiply the matrix of g by the matrix of f or find it directly. Let us work through both methods.

First method : The matrix associated with g is of size 2×4 , denoted by B . Since

$$g(1) = (1, 1), \quad g(x) = (0, 1), \quad g(x^2) = (0, 1) \quad \text{and} \quad g(x^3) = (0, 1),$$

it follows that

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

So,

$$C = BA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}.$$

Second method : To begin, we explicitly find $g \circ f$. Let $(a, b) \in \mathbb{R}^2$. Then

$$(g \circ f)(a, b) = g[f(a, b)] = g(a + b + ax + bx^2 + (a - b)x^3),$$

which simplifies to

$$(g \circ f)(a, b) = (a + b, a + b + a + b + a - b) = (a + b, 3a + b).$$

Since

$$(g \circ f)(1, 0) = (1, 3) \quad \text{and} \quad (g \circ f)(0, 1) = (1, 1),$$

it ensues that

$$C = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}.$$

Exercise 104. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map defined by :

$$f(x, y) = (x - 2y, y).$$

1. Find the matrix A associated with f with respect to the canonical basis of \mathbb{R}^2 , denoted by $\{e_1, e_2\}$.
2. Let $e'_1 = e_1$ and $e'_2 = e_1 + e_2$. Show that $\{e'_1, e'_2\}$, is a new basis of \mathbb{R}^2 .
3. Find the matrix of f in this new basis.

Solution 104.

1. Since

$$f(e_1) = f(1, 0) = (1 - 2 \times 0, 3 \times 0) = (1, 0),$$

and

$$f(e_2) = f(0, 1) = (0 - 2, 3) = (-2, 3),$$

it follows that

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}.$$

2. Given that $\text{card}\{e'_1, e'_2\} = 2 = \dim \mathbb{R}^2$, for e'_1, e'_2 to be a basis, we need only show it is free (linearly independent). However, the vectors $e'_1 = e_1 = (1, 0)$ and $e'_2 = e_1 + e_2 = (1, 1)$ are not proportional, i.e. they are linearly independent. Thus, $\{e'_1, e'_2\}$ is a basis of \mathbb{R}^2 .
3. To find the matrix of f with respect to the new basis $\{e'_1, e'_2\}$, which we denote by B , we compute $f(e'_1)$ and $f(e'_2)$ in terms of e'_1 and e'_2 . We have

$$f(e'_1) = f(e_1) = (1, 0) = e'_1$$

and

$$\begin{aligned} f(e'_2) &= f(e_1 + e_2) = f(e_1) + f(e_2) = (1, 0) + (-2, 3) \\ &= (-1, 3) = -e_1 + 3e_2 = -e'_1 + 3(e'_2 - e'_1) \\ &= -4e'_1 + 3e'_2. \end{aligned}$$

Thus,

$$B = \begin{pmatrix} 1 & -4 \\ 0 & 3 \end{pmatrix}.$$

Exercise 105. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map defined by :

$$f(x, y, z) = (-2z, x + 2y + z, x + 3z).$$

1. Find the matrix associated with f in the canonical basis of \mathbb{R}^3 , noted $\{e_1, e_2, e_3\}$.
2. Let :

$$e'_1 = -e_1 + e_3, \quad e'_2 = e_2, \quad e'_3 = -2e_1 + e_2 + e_3.$$

Prove that $\{e'_1, e'_2, e'_3\}$ is another basis of \mathbb{R}^3 .

3. Check that the matrix associated with f with respect to the basis $\{e'_1, e'_2, e'_3\}$ is diagonal.

Solution 105.

1. Let A be such a matrix. Then

$$f(e_1) = f(1, 0, 0) = (0, 1, 1), \quad f(e_2) = f(0, 1, 0) = (0, 2, 0)$$

and

$$f(e_3) = f(0, 0, 1) = (-2, 1, 3).$$

Thus

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

2. As

$$\det A = -2 \det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = 4 \neq 0$$

$\{e'_1, e'_2, e'_3\}$ forms a linearly independent set, proves that it is a basis of \mathbb{R}^3 .

3. To find the matrix of f with respect to the new basis $\{e'_1, e'_2, e'_3\}$, which we denote by B , we compute $f(e'_1)$, $f(e'_2)$ and $f(e'_3)$ in terms of e'_1 , e'_2 and e'_3 . We have

$$\begin{aligned} f(e'_1) &= f(-e_1 + e_3) = -f(e_1) + f(e_3) = -(0, 1, 1) + (-2, 1, 3) \\ &= (-2, 0, 2) = 2[(-1, 0, 1) + (-1, 0, 0)] \\ &= 2 \underbrace{(e_1 + e_3)}_{=e'_1} \\ &= 2e'_1 \end{aligned}$$

$$\begin{aligned} f(e'_2) &= f(e_2) = (0, 2, 0) = 2 \underbrace{(0, 2, 0)}_{=e'_2} \\ &= 2e'_2 \end{aligned}$$

$$\begin{aligned} f(e'_3) &= f(-2e_1 + e_2 + e_3) = -2f(e_1) + f(e_2) + f(e_3) \\ &= -2(0, 1, 1) + (0, 2, 0) + (-2, 1, 3) \\ &= (-2, 1, 1) = -2(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1) = -2e_1 + e_2 + e_3 \\ &= e'_3 \end{aligned}$$

Thus

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{which is obviously diagonal.}$$

Exercise 106. Consider the matrix :

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix}$$

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear map associated with matrix A with respect to the standard bases (canonical bases) of \mathbb{R}^3 and \mathbb{R}^2 , denoted by $\mathcal{B} = \{e_1, e_2, e_3\}$ and $\mathcal{B}' = \{f_1, f_2\}$ respectively. Let :

$$\begin{cases} e'_1 = e_2 + e_3, \\ e'_2 = e_1 + e_3, \\ e'_3 = e_1 + e_2, \end{cases} \quad \text{and} \quad \begin{cases} f'_1 = \frac{1}{2}(f_1 + f_2), \\ f'_2 = \frac{1}{2}(f_1 - f_2) \end{cases}.$$

1. Show that $\mathcal{B}' = \{e'_1, e'_2, e'_3\}$ is a basis of \mathbb{R}^3 and that $\mathcal{D}' = \{f'_1, f'_2\}$ is a basis of \mathbb{R}^2 .
2. Give the matrix associated with T in this new basis.

Solution 106.

1. Readers may verify for themselves that $\{e'_1, e'_2, e'_3\}$ and $\{f'_1, f'_2\}$ are bases of \mathbb{R}^3 and \mathbb{R}^2 , respectively
2. **First method :** To find the matrix of T with respect to the new bases $\{e'_1, e'_2, e'_3\}$ and $\{f'_1, f'_2\}$ which we denote by B , we compute e_1, e_2, e_3 in terms of e'_1, e'_2, e'_3 and $f(e'_1), f(e'_2), f(e'_3)$ in terms of f'_1, f'_2 .

— We get

$$\begin{cases} f'_1 = \frac{1}{2}(f_1 + f_2), \\ f'_2 = \frac{1}{2}(f_1 - f_2) \end{cases} \implies \begin{cases} f_1 = f'_1 + f'_2, \\ f_2 = f'_1 - f'_2 \end{cases}.$$

— Now we compute

$$\begin{aligned} T(e'_1) &= T(e_2 + e_3) = T(0, 1, 1) = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= (0, -1) = -1f_2 = -(f'_1 - f'_2) \\ &= -f'_1 + f'_2 \end{aligned}$$

$$\begin{aligned} T(e'_2) &= T(e_1 + e_3) = T(1, 0, 1) = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= (3, 0) = 3f_1 = 3(f'_1 + f'_2) \\ &= 3f'_1 + 3f'_2 \end{aligned}$$

$$\begin{aligned} T(e'_3) &= T(e_1 + e_2) = T(1, 1, 0) = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= (1, 5) = f_1 + 5f_2 = (f'_1 + f'_2) + 5(f'_1 - f'_2) \\ &= 6f'_1 - 4f'_2 \end{aligned}$$

Therefore,

$$B = \begin{pmatrix} -1 & 3 & 6 \\ 1 & 3 & -4 \end{pmatrix}.$$

Second method : The transition matrix of \mathcal{B} to \mathcal{B}' is given by :

$$[P]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Also, the transition matrix from \mathcal{D} to \mathcal{D}' is given by :

$$[P]_{\mathcal{D}\mathcal{D}'} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We also know that matrix $[T]_{\mathcal{D}\mathcal{D}'}$ in the new basis is given by the formula

$$[T]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}}.$$

From different ways to find $[P]_{\mathcal{D}'\mathcal{B}'}$, here we choose to express the elements of the basis \mathcal{B}' in terms of the basis \mathcal{D}' , which allows us to determine $[P]_{\mathcal{D}'\mathcal{B}'}$. We have

$$\begin{cases} f'_1 = \frac{1}{2}(f_1 + f_2), \\ f'_2 = \frac{1}{2}(f_1 - f_2) \end{cases} \implies \begin{cases} f_1 = f'_1 + f'_2, \\ f_2 = f'_1 - f'_2 \end{cases}.$$

So

$$[P]_{\mathcal{D}'\mathcal{B}'} = [P]_{\mathcal{B}'\mathcal{D}'}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Thus,

$$[T]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$[T]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[T]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} -1 & 3 & 6 \\ 1 & 3 & -4 \end{pmatrix}.$$

Exercise 107. Let $E = \mathbb{R}_2[x]$ and $f : E \rightarrow E$ be the linear map defined by

$$f[P(x)] = P(x + 1).$$

1. Determine the matrix A associated with f with respect to the standard basis of $\mathbb{R}_2[x]$.
2. Prove that $\left\{1, x, \frac{1}{2}(x^2 - x)\right\}$ is a new basis of $\mathbb{R}_2[x]$.
3. Determine the matrix B associated with f in the new basis.

Solution 107.

1. Since

$$\begin{aligned} f(1) &= 1 = 1 \times 1 + 0 \times x + 0 \times x^2, \\ f(x) &= x + 1 = 1 \times 1 + 1 \times x + 0 \times x^2, \\ f(x^2) &= (x + 1)^2 = x^2 + 2x + 1 = 1 \times 1 + 2 \times x + 1 \times x^2, \end{aligned}$$

it ensues that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Since

$$\text{card} \left\{ 1, x, \frac{1}{2}(x^2 - x) \right\} = 3 = \dim \mathbb{R}_2[x],$$

we know that $\left\{ 1, x, \frac{1}{2}(x^2 - x) \right\}$ is a basis of $\mathbb{R}_2[x]$ once its elements are linearly independent.

Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\alpha + \beta x + \gamma \frac{1}{2}(x^2 - x) = 0.$$

Hence,

$$\alpha + (\beta - \frac{1}{2}\gamma)x + \gamma \frac{1}{2}x^2 = 0.$$

but a polynomial vanishes if all its coefficients vanish, i.e., when $\alpha = \beta = \gamma = 0$. Therefore, $\left\{ 1, x, \frac{1}{2}(x^2 - x) \right\}$ is a basis of $\mathbb{R}_2[x]$.

3. Since

$$\begin{aligned} 1 &= 1 \times 1 + 0 \times x + 0 \times x^2, \\ x &= 0 \times 1 + 1 \times x + 0 \times x^2, \\ \frac{1}{2}(x^2 - x) &= 0 \times 1 - \frac{1}{2} \times x + \frac{1}{2} \times x^2. \end{aligned}$$

the transition matrix from $\{1, x, x^2\}$ to $\left\{ 1, x, \frac{1}{2}(x^2 - x) \right\}$ is given by :

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

while its inverse is given by :

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Consequently,

$$B = P^{-1}AP = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 108. 1. Let f be a linear transformation from \mathbb{R}^3 to \mathbb{R}^3 with a matrix representation in the canonical basis given by :

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}.$$

Give the expression of the matrix B , which represents the matrix A in the new basis of \mathbb{R}^3 , formed by the vectors :

$$e'_1 = e_1 - e_3, \quad e'_2 = e_2 + e_3, \quad \text{and} \quad e'_3 = e_1 + e_3.$$

2. Let $n \in \mathbb{N}$. Find B_n and then deduce A_n .
3. Now, consider three sequences (x_n) , (y_n) , and (z_n) defined by :

$$\begin{cases} x_0 = y_0 = 1, \quad z_0 = 2, \\ x_{n+1} = 3x_n - y_n + z_n, \\ y_{n+1} = 2y_n, \\ z_{n+1} = x_n - y_n + z_n. \end{cases}$$

Find the general terms of x_n , y_n , and z_n .

Solution 108.

1. It is left to the readers to confirm that :

$$B = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

where the transition matrix P and its inverse are given by :

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

respectively.

2. Since B is diagonal, we have for $\forall n \in \mathbb{N}$.

$$B = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}.$$

Since $A = PBP^{-1}$, we can write

$$\begin{aligned} A_n &= (PBP^{-1})^n \\ &= (PBP^{-1})(PBP^{-1}) \dots (PBP^{-1}) \\ &= PB(P^{-1}P)B(P^{-1}P) \dots (P^{-1}P)BP^{-1} \\ &= P \underbrace{B \dots B}_{n \text{ times}} P^{-1} \\ &= PB^n P^{-1}. \end{aligned}$$

After performing calculations, we find that :

$$A_n = \begin{pmatrix} 1 + 2^n & 1 - 2^n & 2^n - 1 \\ 0 & 2 & 0 \\ 2^n - 1 & 1 - 2^n & 1 + 2^n \end{pmatrix}.$$

3. The given system is equivalent to

$$X_{n+1} = AX_n, \quad \forall n \in \mathbb{N},$$

where $X_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$. By induction, we have

$$X_n = AX_{n-1} = A(AX_{n-2}) = A_2X_{n-2} = \cdots = A_nX_0 \quad \forall n \in \mathbb{N}.$$

Thus,

$$X_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = 2^{n-1} \begin{pmatrix} 1 + 2^n & 1 - 2^n & 2^n - 1 \\ 0 & 2 & 0 \\ 2^n - 1 & 1 - 2^n & 1 + 2^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix},$$

which then yields

$$\begin{cases} x_n = 2^{2n}, \\ y_n = 2^n, \\ z_n = 2^n(1 + 2^n), \end{cases} \quad \text{still } \forall n \in \mathbb{N}$$

Exercise 109. Let $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$ be a set of functions from \mathbb{R} to \mathbb{R} . We set

$$E = \text{span} \{e^{-x}, xe^{-x}, x^2e^{-x}\}$$

1. Find $\dim E$.
2. Define $T : E \rightarrow E$ by $T(f) = f'$. Show that T is an endomorphism of E . Determine its matrix A in the basis : $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$.
3. Let :

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- (a) Compute B^n for each $n \in \mathbb{N}$.
- (b) Deduce that :

$$A^n = (-1)^n \left(I - nB + \frac{n(n-1)}{2} B^2 \right), \quad n \in \mathbb{N}.$$

4. Let $n \in \mathbb{N}$. Using the matrix A^n , find the n^{th} derivative of the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by :

$$g(x) = (3 - 2x + 8x^2)e^{-x}.$$

Solution 109.

1. As E is spanned by the functions e^{-x} , xe^{-x} , and x^2e^{-x} , its dimension will be 3 once we establish the linear independence of its elements. Let $a, b, c \in \mathbb{R}$ be such that

$$ae^{-x} + bxe^{-x} + cx^2e^{-x} = 0, \quad \forall x \in \mathbb{R},$$

which simplifies to

$$a + bx + cx^2 = 0, \quad \forall x \in \mathbb{R}.$$

Thus, $a = b = c = 0$, and the set $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$ is a basis of E , thereby $\dim E = 3$.

2. We can easily check that T is linear from E to E , which is then an endomorphism of E . To find the matrix associated with T , we evaluate (for all x) :

$$\begin{aligned} T(e^{-x}) &= -e^{-x} = -1 \times e^{-x} + 0xe - xe^{-x} + 0x^2e^{-x}, \\ T(xe^{-x}) &= e^{-x} - xe^{-x} = 1 \times e^{-x} - 1 \times xe^{-x} + 0 \times 2e^{-x}, \\ T(x^2e^{-x}) &= 2xe^{-x} - x^2e^{-x} = 0 \times e^{-x} + 2 \times xe^{-x} - 1 \times x^2e^{-x}. \end{aligned}$$

Hence,

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

3. (a) It is easy to see that

$$B^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{So, } B^n = 0_{\mathcal{M}_3(\mathbb{R})} \forall n \geq 3.$$

- (b) First, observe that $A = -I + B$. We have two possible approaches : Applying the binomial formula for I and B or proving the result by induction, given that we already know the expected form of A^n . However, we opt for an inductive proof. The base case follows from the identity

$$A^2 = I - 2IB + B^2,$$

which holds due to the commutativity of I and B . Now, assume the formula for A^n holds. Then, and as $B^3 = 0_{\mathcal{M}_3(\mathbb{R})}$,

$$\begin{aligned} A^{n+1} &= A^n A \\ &= (-1)^n \left(I - nB + \frac{n(n-1)}{2} B^2 \right) (-I + B) \\ &= (-1)^n \left(-I + nB - \frac{n(n-1)}{2} B^2 + B - nB^2 \right) \\ &= (-1)^{n+1} \left(I - (n+1)B + \frac{n(n+1)}{2} B^2 \right) \end{aligned}$$

, Thus, we have shown the formula for A^n . The matrix A^n is then written for all n as :

$$A^n = (-1)^n \begin{pmatrix} 1 & -n & n(n-1) \\ 0 & 1 & -2^n \\ 0 & 0 & 1 \end{pmatrix}.$$

4. First, we should refrain from using the Leibniz formula, even though it is applicable to the functions $x \rightarrow e^{-x}$ and $x \rightarrow 3 - 2x + 8x^2e^{-x}$, since another method has been specified. Now, we can write :

$$g' = T(g), \quad g'' = T(g') = (T \circ T)(g), \dots,$$

and

$$g^{(n)} = \underbrace{(T \circ T \circ \dots \circ T)}_{n \text{ times}}(g).$$

So, the matrix of T^n corresponds to A^n . Thus, the matrix of $g^{(n)}$ is given by :

$$(-1)^n \begin{pmatrix} 1 & -n & n(n-1) \\ 0 & 1 & -2n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix} = (-1)^n \begin{pmatrix} 8n^2 - 6n + 3 \\ -16n - 2 \\ 8 \end{pmatrix}.$$

Accordingly,

$$g^{(n)}(x) = (-1)^n e^{-x} (8n^2 - 6n + 3 - (16n + 2)x + 8x^2) \text{ for } n = 1, 2, \dots$$

Exercise 110. Use Cramer's rule to solve for x in the system :

$$\begin{cases} 2x + 3y = 8 \\ 4x - y = 2 \end{cases}$$

Solution 110. $A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$, $\det A = 2(-1) - 3(4) = -2 - 12 = -14$. Replace first column with $\mathbf{b} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$: $A_x = \begin{pmatrix} 8 & 3 \\ 2 & -1 \end{pmatrix}$, $\det A_x = 8(-1) - 3(2) = -8 - 6 = -14$. Then

$$x = \frac{\det A_x}{\det A} = \frac{-14}{-14} = 1.$$

Exercise 111. Solve the linear system using Gaussian elimination :

$$\begin{cases} x + 2y - z = 1 \\ 2x + 3y + z = 2 \\ 3x + 5y + 0z = 3 \end{cases}$$

Solution 111. Write the augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 5 & 0 & 3 \end{array} \right]$$

Perform row operations :

$$R_2 \leftarrow R_2 - 2R_1 : \quad (0 \quad -1 \quad 3 \quad 0)$$

$$R_3 \leftarrow R_3 - 3R_1 : \quad (0 \quad -1 \quad 3 \quad 0)$$

Now we have :

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & -1 & 3 & 0 \end{array} \right]$$

Subtract R_2 from R_3 : $R_3 \leftarrow R_3 - R_2$ gives zero row. So the system reduces to :

$$\begin{cases} x + 2y - z = 1 \\ -y + 3z = 0 \end{cases}$$

We get

$$y = 3z \text{ and } x = 1 - 5z.$$

So solution :

$$(x, y, z) = (1 - 5t, 3t, t) \quad \text{for any } t \in \mathbb{R}.$$

Exercise 112. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ if it exists.

Solution 112. Compute determinant : $\det A = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2 \neq 0$, so inverse exists. Using formula for 2×2 :

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}.$$

Alternatively,

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Exercise 113. Compute the determinant of

$$B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Solution 113. Expand along first row :

$$\begin{aligned} \det B &= 1 \cdot \det \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} \\ &= 1[(3)(0) - (1)(1)] + 2[(-1)(1) - (3)(2)] = -1 + 2(-1 - 6) = -1 + 2(-7) = -1 - 14 = -15. \end{aligned}$$

Exercise 114. Determine whether the set of matrices

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{R}) : a + d = 0 \right\}$$

is a subspace of $M_{2 \times 2}(\mathbb{R})$.

Solution 114. Check :

— Zero matrix has $0 + 0 = 0$, so $\mathbf{0} \in S$.

— If $A, B \in S$, then

$$(a_{11} + a_{22}) = 0 \quad \text{and} \quad (b_{11} + b_{22}) = 0.$$

For $A + B$, the sum of diagonal entries is

$$(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0 + 0 = 0,$$

so $A + B \in S$.

— For scalar c , diagonal entries of cA are ca_{11} and ca_{22} ,

$$ca_{11} + ca_{22} = c(a_{11} + a_{22}) = c \cdot 0 = 0, \quad \text{so} \quad cA \in S.$$

Thus S is a subspace.

Exercise 115. Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

Solution 115. Observe that the second row is twice the first, and the third row is three times the first. Thus all rows are multiples of the first, so the row space is one-dimensional. Hence $\text{rank } A = 1$.

Exercise 116. Solve the system using matrix inversion :

$$\begin{cases} 2x + y = 5 \\ x - 3y = 6 \end{cases}$$

Solution 116. In matrix form : $A\mathbf{x} = \mathbf{b}$ with $A = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

Compute A^{-1} : $\det A = 2(-3) - 1(1) = -6 - 1 = -7$. Then

$$A^{-1} = \frac{1}{-7} \begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}.$$

Then

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} \frac{3}{7} \cdot 5 + \frac{1}{7} \cdot 6 \\ \frac{1}{7} \cdot 5 - \frac{2}{7} \cdot 6 \end{pmatrix} = \begin{pmatrix} \frac{15+6}{7} \\ \frac{5-12}{7} \end{pmatrix} = \begin{pmatrix} \frac{21}{7} \\ -\frac{7}{7} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Exercises without Solutions

Exercise 117. Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ where $\forall v \in \mathbb{R}^2$, $T(v) = Av$

a) : Compute $T \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **b)** : Find v such that $T(v) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ **c)** : Is $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ in the image of T

Exercise 118. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear map such that

$$T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

a) : Compute $T \begin{pmatrix} 8 \\ 13 \\ 18 \end{pmatrix}$ **b)** : Compute the dimension of T

Exercise 119. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear map defined as

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + 4y \\ x + \alpha y \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ is a parameter

1. Find the matrix $[T]$ associated to T with respect to the standard bases of \mathbb{R}^2 and \mathbb{R}^3 .
2. For which values of α is T one-to-one (injective) ?
3. For which values of α is T onto (surjective) ?

Exercise 120. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map such that

$$T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 18 \end{pmatrix}$$

1. Compute the dimensions of $\ker(T)$ and $\text{Im}(T)$
2. Compute $T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise 121. Let V be the real vector space $\mathbb{R}_2[x]$, whose elements are the polynomials with real coefficients and degree less than or equal to two. Consider the linear map

$$L : V \rightarrow V, \quad L(ax^2 + bx + c) = (3a + 3b)x^2 + (b + c)x + (a + b + 2c).$$

1. Compute the dimensions of the kernel and the image of L
2. Decide if L is invertible and, if so, compute the image of the polynomial $4x^2 + x + 1$ through the inverse function.

Exercise 122. Let $\mathbb{Q}_2[x]$ be the space of polynomials with rational coefficients and degree less than or equal to 3 and let $L : \mathbb{Q}_2[x] \rightarrow \mathbb{Q}_2[x]$ be the linear map given by

$$L(ax^3 + bx^2 + cx + d) = (a + b + c)x^3 + dx^2 + 2c.$$

Compute a basis of $\ker(L)$, a basis for $\text{Im}(L)$, and a basis of $\text{Im}(L \circ L \circ L)$.

Exercise 123. Let $L_A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear map defined as $L_A(v) = Av$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 3 & 2 & -1 \end{pmatrix}$$

1. Find a basis of $\ker L_A$ and complete it to a basis of \mathbb{R}^4
2. Find a basis of $\text{Im}(L_A)$.

3. Does the vector $b = \begin{pmatrix} 1 \\ 8 \\ -3 \\ -2 \end{pmatrix}$ belong to $\text{Im}L_A$?

Exercise 124. Consider the linear map $\phi : \mathcal{M}_2(\mathbb{R}) \rightarrow \mathcal{M}_2(\mathbb{R})$ such that, for any matrix

$A \in \mathcal{M}_2(\mathbb{R})$, we have $\phi(A) = AB$, where B is the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

1. Determine the dimension and a basis of $\ker(\phi)$
2. Determine the dimension and a basis of $\text{Im}(\phi)$

Exercise 125. The standard matrix of T is given below.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 5 & 3 \\ 3 & 2 & 3 \end{pmatrix}$$

Is the linear map T One-to-one? Onto?

Exercise 126. what value of x will make A and B commute?

$$A = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Exercise 127. Let

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, T(x, y, z) = (x + 2y - z, -2x - 4y + 2z)$$

Are $(3, -2, -1) \in \ker(T)$, $(1, 1, 2) \in \ker(T)$, $(2, -6) \in \text{Im}(T)$, $(1, 2) \in \text{Im}(T)$

Exercise 128. Compute $\ker(T)$, $\text{Im}(T)$, $\text{Nullity}(T)$ and $\text{rank}(T)$

1. $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, T_1(x, y) = (x + y, y, 0)$.
2. $T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, T_2(x, y, z) = (x - y, y + z, 0)$
3. $T_3 : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_1[x], T_3(a + bx + cx^2) = a(x + 1) + cx + b$.
4. $T_4 : \mathbb{R}_2[x] \longrightarrow \mathbb{R}, T_4(P(x)) = P(2) - P(1)$.
5. $T_5 : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathbb{R}_3[x], T_5 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + (b - c)x + ax^2$.

Exercise 129. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, T(x, y, z) = (2x + y - z, 3x - y + 2z)$. Find the standard matrix for T

Exercise 130. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^2, T(x, y, z, t) = \begin{pmatrix} 5 & 3 & 8 & 2 \\ 1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$.

1. Find basis for $\text{Im}(T)$
2. Find basis for $\ker(T)$
3. Find $\text{rank}(T)$ and $\text{Nullity}(T)$.

Exercise 131. Let a basis $\mathcal{B} = \left\{ \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$. If $[v]_{\mathcal{E}} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$ find $[v]_{\mathcal{B}}$, where \mathcal{E} is a standard basis of \mathbb{R}^2 .

Exercise 132. Let $T_i : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, where

$$T_1(x, y, z) = (x + y, x - z), \quad T_2(x, y, z) = (x + y - z, x).$$

Compute $T_1 \circ T_2$

Exercise 133.

Consider the basis $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$ for \mathbb{R}^3 and let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear

map such that $T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $T \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $T \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -9 \\ 6 \end{pmatrix}$. Find $T(X)$, $X \in \mathbb{R}^3$.

Find $T \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Exercise 134. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear map such that

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + 4y \\ x + \alpha y \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ is a parameter

- (a) Find the matrix $[f]$ associated to f with respect to the standard bases of \mathbb{R}^2 and \mathbb{R}^3 .
- (b) For which values of α is f (one-to-one) injective?
- (c) For which values of α is f (onto) surjective?

Exercise 135. Compute the inverse of A where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{pmatrix}$

Exercise 136. Let $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$. Use A^{-1} to find v where $Av = b$

Exercise 137. Find $\det(A) = \begin{pmatrix} 1 & 7 & 2 & 5 \\ 0 & 8 & 4 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 9 & 3 & 0 \end{pmatrix}$

Exercise 138. Find : $\det \begin{pmatrix} 1 & 7 & 9 & 12 & 16 \\ 0 & 2 & 0 & 5 & 26 \\ 0 & 0 & 2 & 4 & 25 \\ 0 & 0 & 0 & 3 & 50 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$, $\det \begin{pmatrix} -1 & 2 & 3 & 1 & -2 \\ 0 & 2 & 0 & 5 & 2 \\ 0 & 0 & 2 & 4 & 7 \\ 0 & 0 & 0 & 3 & 5 \\ -2 & 4 & 6 & 2 & -4 \end{pmatrix}$

Exercise 139. Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{pmatrix}$$

1. Determine if the system $Ax = 0$ has zero, one or infinitely many solutions, and compute a basis of the space of solutions.
2. Is it true that the system $Ax = b$ has a solution for any $b \in \mathbb{R}^3$? If so, prove the statement, otherwise find a counterexample.

Exercise 140. Determine the number of solutions of the following system

$$\begin{cases} x + 2y - 3z = 4 \\ 4x + y + 2z = 6 \\ x + 2y + (a^2 - 19)z = a \end{cases}$$

depending on the parameter $a \in \mathbb{R}$.

Exercise 141. Find a matrix A where $Col(A) = \left\{ \begin{pmatrix} 7a + 5b \\ a - b \\ 2b \end{pmatrix}; a, b \in \mathbb{R} \right\}$

Exercise 142. Find a basis for $Col(A)$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$

Exercise 143. Let $T : \mathbb{R}_2[x] \rightarrow \mathbb{R}_1[x]$, where $T(a + bx + cx^2) = a + b(x + 1)$. Determine whether T is linear or not

Exercise 144. Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$, compute :

1. $A + B$
2. $2A - 3B$
3. AB and BA
4. $A^T + B^T$

Exercise 145. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$. Compute AB and BA if possible.

Explain why one product might not be defined.

Exercise 146. Solve the system using Gaussian elimination :

$$\begin{cases} x + 2y + 3z = 1 \\ 2x + 5y + 7z = 2 \\ 3x + 7y + 10z = 3 \end{cases}$$

Solution 117. Write the augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & 2 \\ 3 & 7 & 10 & 3 \end{array} \right]$$

Perform row operations :

$$\begin{aligned} R_2 &\leftarrow R_2 - 2R_1 : & (0 \quad 1 \quad 1 \quad 0) \\ R_3 &\leftarrow R_3 - 3R_1 : & (0 \quad 1 \quad 1 \quad 0) \end{aligned}$$

Thus we have :

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

Subtract R_2 from R_3 : $R_3 \leftarrow R_3 - R_2$ gives zero row. The system reduces to :

$$\begin{cases} x + 2y + 3z = 1 \\ y + z = 0 \end{cases}$$

From the second, $y = -z$. Substitute into first : $x + 2(-z) + 3z = 1 \Rightarrow x + z = 1 \Rightarrow x = 1 - z$. So the solution is $(x, y, z) = (1 - t, -t, t)$ for any $t \in \mathbb{R}$.

Exercise 147. Solve each system by Gaussian elimination.

1.

$$\begin{cases} 2x + 3y = 5 \\ 4x + 6y = 10 \end{cases}$$

2.

$$\begin{cases} x - 2y + z = 4 \\ 2x + y - z = 1 \\ 3x + 3y - 2z = 2 \end{cases}$$

3.

$$\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 7 \\ 3x + 2y - z = 4 \end{cases}$$

Exercise 148. For each system, determine if A is invertible and solve using the inverse method.

1.
$$\begin{cases} 3x + y = 8 \\ 2x - 5y = 11 \end{cases}$$

2.
$$\begin{cases} x + 2y + z = 4 \\ 2x + 3y + z = 7 \\ x + y + 2z = 5 \end{cases}$$

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